Liouville skeleta and homological mirror symmetry for unimodular acyclic cluster models

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1 The coefficient-free case of type A_{2n}

1.1 The full affine variety and its integral node charts

Let $n \geq 1$ and put $m = 2n$. We use the coefficient-free acyclic seed of type A_m, with every vertex mutable and with bipartite orientation

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow \cdots .$$

Our exchange-matrix convention is

$$\varepsilon_i = (-1)^{i+1}, \qquad B_{ij} = \begin{cases} \varepsilon_i, & |i-j| = 1, \\ 0, & \text{otherwise;} \end{cases}$$

thus $B_{ij} > 0$ records an arrow from i to j. Put $x_0 = x_{m+1} = 1$. The full affine cluster variety is

$$X_m = \operatorname{Spec} \frac{\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m]}{(x_i y_i - 1 - x_{i-1}x_{i+1})_{i=1}^m}. \tag{1.1}$$

Here y_i is the variable obtained by mutating x_i once in this seed. We retain the entire affine spectrum, including all simultaneous-zero strata. There are no frozen variables in this example. The initial torus

$$T = \{x_1 \cdots x_m \neq 0\} \subset X_m$$

is identified with $(\mathbb{C}^*)^m$ by $y_i = (1 + x_{i-1}x_{i+1})/x_i$.

The matrix B is skew-symmetric. Expansion at an endpoint gives

$$\det B_m = \det B_{m-2}, \qquad \det B_0 = 1,$$

so $\det B = 1$. Thus the even path satisfies the unimodularity assumption used in the general construction.

Write $\operatorname{Ind}(P_m)$ for the subsets of $\{1, \ldots, m\}$ containing no adjacent vertices. For $I \in \operatorname{Ind}(P_m)$ put $J = \{1, \ldots, m\} \setminus I$ and

$$r_i = x_{i-1}x_{i+1}, \qquad U_I = \{x_j \neq 0 \text{ for every } j \in J\} \subset X_m.$$

These opens cover X_m: if $x_i = 0$, the exchange equation gives $r_i = -1$, so no neighbor of i can vanish. Eliminating y_j for $j \in J$ gives

$$\mathcal{O}(U_I) = \frac{\mathbb{C}[x_i, y_i \ (i \in I), x_j^{\pm 1} \ (j \in J)]}{(x_i y_i - 1 - r_i)_{i \in I}}. \tag{1.2}$$

The exponent vectors

$$c_i = \sum_{j \sim i} e_j \in \mathbb{Z}^J, \qquad i \in I,$$

are signed rows of B. Since B is unimodular, they generate a primitive sublattice of $\mathbb{Z}^m$. Independence of I puts this sublattice inside $\mathbb{Z}^J$, where it remains primitive: a vector of $\mathbb{Z}^J$ whose nonzero integer multiple lies in the sublattice already lies there by primitivity in $\mathbb{Z}^m$. Complete the c_i to an integral basis of $\mathbb{Z}^J$. The corresponding Laurent coordinates

$$(r_i)_{i \in I}, \quad w_1, \ldots, w_{m-2|I|}$$

identify (1.2) with the product

$$U_I \cong \mathcal{N}^{|I|} \times (\mathbb{C}^*)^{m-2|I|}, \qquad \mathcal{N} = \{(u, v, q) : uv = 1 + q, \ q \neq 0\} \cong \mathbb{C}^2 \setminus \{uv = 1\}. \tag{1.3}$$

The inverse uses Laurent monomials and the regular expressions for the eliminated partner coordinates. In particular, these are integral algebraic charts and require no finite cover. Directly from their nonvanishing conditions,

$$U_I \cap U_{I'} = U_{I\cap I'}.$$

Setting $x_i = 0$ for $i \in I$ in these charts gives

$$\{x_i = 0 \iff i \in I\} \cong \mathbb{A}^{|I|} \times (\mathbb{C}^*)^{m-2|I|}.$$

Consequently X_m is smooth of complex dimension $m = 2n$ and real dimension $2m = 4n$. The initial torus is dense in every product chart, so X_m is irreducible.

1.2 The symplectic form and the cap attachments

On T set

$$\Omega = \frac{1}{2}\sum_{i,j} B_{ij}\, d\log x_i \wedge d\log x_j, \qquad \omega = \operatorname{Im}\Omega. \tag{1.4}$$

On U_I, the incident-edge terms at $i \in I$ combine as

$$\varepsilon_i\, d\log x_i \wedge d\log r_i = \varepsilon_i \frac{dx_i \wedge dy_i}{x_i y_i - 1}.$$

The denominator is invertible. Because I is independent, these cancellations take place simultaneously without counting any edge twice; all remaining terms involve invertible coordinates. This extends Ω holomorphically. Its top exterior power is a nonzero constant times

$$\bigwedge_{i\in I} \frac{dx_i \wedge dy_i}{r_i} \wedge \bigwedge_{\alpha=1}^{m-2|I|} d\log w_\alpha.$$

It is nowhere zero, so the extended form is holomorphic symplectic. The extensions agree because they agree on the dense initial torus.

The torus coordinates $x_i = e^{A_i + \sqrt{-1}\theta_i}$ give

$$\omega = \sum_{i,j} B_{ij}\, dA_i \wedge d\theta_j, \qquad \lambda_0 = \sum_{i,j} A_i B_{ij}\, d\theta_j.$$

Thus $p_j = \sum_i A_i B_{ij}$ identifies this torus region with $T^*(\mathbb{R}/2\pi\mathbb{Z})^m$ and identifies λ_0 with $\sum_j p_j\, d\theta_j$.

A Liouville primitive is a real one-form α with $d\alpha = \omega$; its vector field is defined by $\iota_Z \omega = \alpha$. Here the skeleton means the set of points whose forward Liouville orbits have compact closure. Finite type means that the manifold is the completion of a compact Liouville domain. Thus its smooth Lagrangian strata have real dimension m.

Define the compact cap set

$$K_m = \{(x, y) \in X_m : |x_i| \le 1,\ (1 - |x_i|^2) y_i = 0 \text{ for all } i\}. \tag{1.5}$$

For $I \in \operatorname{Ind}(P_m)$ its closed cap block is

$$P_I = \{y_i = 0,\ |x_i| \le 1\ (i \in I); \quad |x_j| = 1\ (j \notin I)\} \subset X_m. \tag{1.6}$$

The exchange equations impose $r_i = -1$ for $i \in I$. Primitivity of these characters makes their common level in $(S^1)^J$ a connected torus of dimension $m - 2|I|$. The disk coordinates x_i, $i \in I$, are free. Hence

$$P_I \cong (D^2)^{|I|} \times (S^1)^{m-2|I|}. \tag{1.7}$$

This product splitting is noncanonical, whereas the realization (1.6) in the original coordinates is fixed. If $|x_i| < 1$ at a point of K_m, then $y_i = 0$ and $r_i = -1$; since every modulus is at most one, every neighbor of i has modulus one. The set of such indices is therefore independent, and

$$K_m = \bigcup_{I \in \mathrm{Ind}(P_m)} P_I. \tag{1.8}$$

The central block is $P_\varnothing = (S^1)^m$. Write $x_j = e^{\sqrt{-1}\theta_j}$ there, put $\theta_0 = \theta_{m+1} = 0$, and set

$$H_i = \{\theta_{i-1} + \theta_{i+1} = \pi \pmod{2\pi}\}, \qquad \gamma_i = e_i \in H_1((S^1)^m; \mathbb{Z}).$$

For independent I, the intersection $H_I = \bigcap_{i \in I} H_i$ is a connected torus of dimension $m - |I|$. The circles γ_i, $i \in I$, lie in its tangent lattice. Filling these circles by the disk coordinates x_i gives P_I. The face $|x_i| = 1$ of P_I maps into $P_{I \setminus \{i\}}$ by the identity on all original coordinates, with the additional equation $y_i = 0$. All multiple faces use these same coordinate restrictions, so their maps agree on every intersection.

For example, in type A_4 the double cap indexed by $\{1, 3\}$ satisfies $x_2 = -1$ and $x_2 x_4 = -1$, hence $x_4 = 1$. Its full parametrization is

$$(x_1, x_2, x_3, x_4) = (u, -1, v, 1), \qquad (y_1, y_2, y_3, y_4) = (0, -1 - uv, 0, 1 + v), \qquad |u|, |v| \le 1.$$

The two solid-torus faces map to the two singleton caps, and their common torus maps to $H_1 \cap H_3$. In particular, the filling equations are the original character equations $r_i = -1$; a residual torus coordinate obtained after another filling need not itself have value -1.

1.3 The Liouville primitive

We specify the primitive before stating the categorical result. Put

$$P_i = \begin{cases} r_i & i \text{ odd}, \\ 1 & i \text{ even}, \end{cases} \qquad Q_i = \begin{cases} 1 & i \text{ odd}, \\ r_i & i \text{ even}, \end{cases} \qquad q_i = P_i/Q_i = r_i^{\varepsilon_i}.$$

Thus $x_i y_i = P_i + Q_i$. The symbol r_i always denotes the unsigned neighbor product, whereas q_i is the signed exchange character used in the general construction. Near $x_i = 0$, both neighbors are units, so Q_i is invertible and $\widetilde{y}_i = y_i/Q_i$ is regular. The normalized equation is $x_i \widetilde{y}_i = 1 + q_i$.

On the initial torus write

$$A_i = \log|x_i|, \quad \theta_i = \arg x_i, \quad S_i = (BA)_i = \log|q_i|, \quad T_i = \arg(-q_i).$$

Angles have period 2π. Choose $\epsilon > 0$ and smooth functions as follows. The function $\chi : \mathbb{R} \to [0, 1]$ is one on $(-\infty, -\epsilon]$ and zero on $[-\epsilon/2, \infty)$; ρ is compactly supported, takes values in $[0, 1]$, and equals one near zero. The real periodic function ϑ is supported in a small arc about zero, equals T near zero, and satisfies $\vartheta'(T) \le 1$ and $T\vartheta(T) \ge 0$ in that arc. For a sufficiently large constant $\kappa > 0$, choose h with

$$h \le 0, \quad 0 \le h' \le 1, \quad h(A) > A \ (A < 0), \quad h(A) = 0 \ (A \ge -2\epsilon), \quad h(A) = A + \kappa \ (A \ll 0).$$

The construction and estimates for these choices are given in Section 2.2. Define

$$\chi_i(A) = \prod_{\substack{1 \le j \le m \\ |j-i|=1}} \chi(A_i - A_j), \qquad H = \sum_{i=1}^{m} h(A_i)\chi_i(A)\rho(S_i)\vartheta(T_i),$$
$$\alpha = A^{\mathsf{T}} B \, d\theta - dH. \tag{1.9}$$

All cutoffs are real functions; the periodic angular cutoff makes H single-valued on the entire initial torus.

Proposition 1.1. *The form (1.9) extends smoothly to X_m and satisfies $d\alpha = \omega$. Its Liouville vector field Z, defined by $\iota_Z\omega = \alpha$, is complete in both directions and of finite type. Its full skeleton is K_m in (1.5). Every smooth Lagrangian component of this skeleton admits a proper exact generalized cocore.*

Proof. We first check extension and the skeleton in the displayed coordinates. At a zero set I, independence implies that $\chi_i = 1$ for $i \in I$ near that stratum and that all corrections at their neighbors vanish. With the branch near 1, put

$$L_i = \operatorname{Log}(1 - x_i\widetilde{y}_i) = S_i + \sqrt{-1}T_i.$$

In a sufficiently small neighborhood the singular normal contribution to α is

$$A_i\,dT_i - S_i\,d\theta_i - d((A_i + \kappa)T_i) = -\operatorname{Im}\left(L_i\frac{dx_i}{x_i} + \kappa\,dL_i\right).$$

Since $L_i/x_i = -\widetilde{y}_i + O(x_i)$, this expression is smooth. The other terms use only unit coordinates. This proves simultaneous extension at every independent zero set. The equality $d\alpha = \omega$ then extends from the dense torus.

Direct contraction gives

$$ZA_i = A_i - h(A_i)\chi_i\rho(S_i)\vartheta'(T_i), \qquad BZ\theta = \partial_A H.$$

In particular,

$$A_i > 0 \implies ZA_i = A_i > 0, \qquad A_i < 0 \implies ZA_i \le A_i - h(A_i) < 0.$$

All derivatives of the cutoffs are bounded, and h has at most linear growth, so $|ZA| + |Z\theta| \le C(1 + |A|)$ on the torus. Gronwall's inequality excludes finite-time escape there. On $x_i = 0$, the normal equation is

$$Z\widetilde{y}_i = (1 + \kappa - \sqrt{-1}\nu_i)\widetilde{y}_i$$

for a smooth real angular coefficient ν_i. The nonzero logarithmic coordinates retain the same linear growth bound. The zero strata are invariant, so the argument proves completeness on all of X_m, as detailed in Lemma 2.5.

Suppose a forward orbit is relatively compact. It has no positive logarithmic radius. Every initially negative radius tends to $-\infty$; otherwise the strict differential inequality would rule out its finite negative limit. Together with the initially zero coordinates these indices form an independent set I, by taking a convergent subsequence and using the exchange equations. All their neighbors consequently have modulus one along the orbit. For $i \in I$ with $x_i \neq 0$, boundedness of y_i forces $q_i \to -1$. In the eventual deep-radius region, $ZT_i = \vartheta(T_i)$; near zero this is $ZT_i = T_i$. Convergence to zero therefore forces $T_i = 0$. Uniqueness for the angular differential equation propagates this identity backward, so $y_i = 0$ on the whole orbit. If $x_i = 0$ initially, the displayed normal equation and boundedness force $y_i = 0$ as well. Every trapped point lies in K_m.

Conversely, the cap equations imply that the indices with $|x_i| < 1$ are independent, their neighbors have modulus one, and their exchange characters equal -1. The angular motion vanishes, the negative logarithmic radii decrease, and $y_i = 0$ persists. Thus K_m is invariant. It is compact: $|x_i| \le 1$, while $y_i = 0$ for $|x_i| < 1$ and $|y_i| \le 2$ for $|x_i| = 1$. Hence every point of K_m is trapped.

For finite type one also needs control of the negative flow. The backward estimates of Lemma 2.7 apply with this path matrix: every compact set has compact backward saturation and approaches K_m uniformly. Proposition 2.8 constructs from these estimates a proper smooth function F with $ZF > 0$ outside a compact set. A large regular level bounds a

Liouville domain whose completion is X_m. These arguments do not require any assumption beyond the acyclicity and unimodularity already verified here.

For clarity, the cocores have an explicit description in the same coordinates. Given independent I, put $R_I = \{1, \dots, m\} \setminus (I \cup N(I))$. Fix the phases of x_j, $j \notin I$, so that $q_i = -1$ in phase for $i \in I$, and so that $\arg(-q_j)$ is outside the support of ϑ for $j \in R_I$. Primitivity of the corresponding rows of B permits these choices. Then set

$$C_I = \Big\{ x_i = 0,\ y_i \in \mathbb{C}\ (i \in I); \quad x_j = e^{A_j + \sqrt{-1}\theta_j}\ (j \notin I), \quad B_{I,I^c} A_{I^c} = 0 \Big\}.$$

All other y_j are determined by the exchange equations. This is $\mathbb{C}^{|I|} \times \mathbb{R}^{m-2|I|}$, and $\alpha|_{C_I} = 0$. Proposition 2.9 proves properness directly from the independent-zero-set condition, proves cylindricality using the finite-type end, and shows that C_I meets the smooth part of the I-cap transversely in exactly one point. Its construction covers every smooth Lagrangian component. □

1.4 The wrapped category

All categories below are $\mathbb{C}$-linear enhanced categories. We write $\operatorname{Perf} \mathcal{W}$ for the pretriangulated idempotent completion of the wrapped Fukaya category, and $\mu \operatorname{Sh}^w$ for the compact objects in the large microlocal category. Compactness here is categorical; it does not mean finite-dimensional microstalks. The grading and orientation data are those induced by the stable polarization of Proposition 7.4. Its restriction to K_m is the toric–cap Maslov datum η of Proposition 5.4. These choices, and their compatibility, are part of the setup of the equivalence.

Theorem 1.2. *For $m = 2n$ and the primitive above there are equivalences*

$$\operatorname{Perf} \mathcal{W}(X_m, \alpha)^{\mathrm{op}} \simeq \mu \operatorname{Sh}^w_{K_m}(K_m; \eta) \simeq \operatorname{Perf}(X_m) = D^b \operatorname{Coh}(X_m).$$

If $\iota : X_m \hookrightarrow \mathbb{C}^N$ is any closed algebraic embedding and

$$\beta = \frac{1}{2} \iota^* \operatorname{Im} \left(\sum_{k=1}^{N} \overline{z}_k \, dz_k \right),$$

then, with transported grading and orientation data,

$$\operatorname{Perf} \mathcal{W}(X_m, \beta)^{\mathrm{op}} \simeq \operatorname{Perf}(X_m).$$

Proof. We give the algebraic diagram explicitly for the even path. Let

$$X_\Sigma = \mathbb{A}^m \setminus \bigcup_{i=1}^{m-1} V(z_i, z_{i+1}), \qquad D_i = \{z_i = 0\} \subset X_\Sigma,$$

with $z_0 = z_{m+1} = 1$. Every neighbor of i is a unit on D_i. The center for its cap is therefore the smooth divisor

$$H_i^{\mathrm{alg}} = \{ z_i = 0,\ 1 + z_{i-1} z_{i+1} = 0 \} \subset D_i.$$

Normalize each (J, C)-vertex by the shift $s(J, C) = -|J| - |C|$, as in Section 6.2. The two functors in the resulting elementary corestriction diagram are

$$\operatorname{Perf}(X_\Sigma) \xleftarrow{Ri_*} \operatorname{Perf}(D_i) \xrightarrow{Lj^*} \operatorname{Perf}(H_i^{\mathrm{alg}}).$$

Before this normalization, the left adjoint of the normal microstalk Li^* is $Ri_*[-1]$; the vertex shifts account for the displayed unshifted Ri_*. The normal sheaf functor is the Koszul microstalk for z_i; the disk-boundary functor specializes the boundary monodromy to one. With the cap

marking and the sign rescaling of Proposition 5.4, this is the full derived specialization at $q_i = -1$. For even i, replacing $1 + r_i^{-1}$ by $1 + r_i$ multiplies the equation by a unit on the normal stratum, with its specified Koszul unit-change map.

Adjacent divisors are disjoint. On an independent intersection the characters r_i are independent primitive Laurent coordinates, so the centers are transverse coordinate pairs $(z_i, 1 + r_i)$. Consequently all multiple cap diagrams are the independent cubes, with the ordered Koszul tensor comparisons and their higher coherences. The open-cover descent calculation in Sections 6.1–6.2 applies to the actual marked germ of the cap support. For each divisor and center the blowup/delete pushout of [2, Theorem 4.13] identifies the elementary colimit with the affine modification

$$z_i w_i = 1 + z_{i-1} z_{i+1}.$$

Applying these pushouts in any fixed order, and retaining every independent multiple intersection, gives precisely the charts (1.2). They cover the full X_m. Proposition 6.3 identifies the entire coherent diagram and its colimit, including its higher transformations; hence $\mu \operatorname{Sh}^{w}_{K_m}(K_m; \eta) \simeq \operatorname{Perf}(X_m)$.

The remaining comparison is global. The primitive vanishes on the strata of K_m and is locally an analytic primitive plus dG with $G|_{K_m} = 0$, by the extension formula above. Lemmas 7.1–7.2 therefore give the required analytic contactification of the core. Proposition 7.4 supplies a compatible stable polarization, and the explicit cocores above are homological cocores by Proposition 7.5. The GPS comparison is fully faithful and sends these cocores to co-representatives of microstalks, which generate the compact microlocal category. It is thus an equivalence after pretriangulated idempotent completion, exactly as proved in Theorem 7.6. This proves the first assertion. Smoothness of X_m gives $\operatorname{Perf}(X_m) = D^b \operatorname{Coh}(X_m)$.

Finally, the quadratic Poisson embedding and primitive estimates of Propositions 3.5–3.6 apply to the path presentation. The complete Moser comparison of Theorem 3.1 and the four-domain transport of Theorem 4.1 give the wrapped equivalence with an affine Kähler structure. Lemma 3.10 makes this conclusion independent of the closed algebraic embedding, as summarized in Corollary 4.7. □

The opposite category is the convention of the GPS comparison. Applying $\operatorname{RHom}(-, \mathcal{O}_{X_m}) : \operatorname{Perf}(X_m)^{\mathrm{op}} \simeq \operatorname{Perf}(X_m)$ gives the corresponding covariant formulation.

2 General acyclic cluster models

Let $B = (B_{ij})$ be an integral skew-symmetric $d \times d$ matrix with $\det B = 1$. Partition its index set $\{1, \dots, d\}$ into mutable and frozen vertices, $M \sqcup F$. Our arrow convention is that $B_{ij} > 0$ records B_{ij} arrows from i to j. Assume that the quiver induced on M is acyclic; multiple arrows are permitted. For $i \in M$ put

$$P_i = \prod_{B_{ij} > 0} a_j^{B_{ij}}, \qquad Q_i = \prod_{B_{ij} < 0} a_j^{-B_{ij}},$$

where an empty product is 1, and consider the explicitly presented affine variety

$$Y_B = \{a_i b_i = P_i + Q_i \ (i \in M), \quad a_f \neq 0 \ (f \in F)\}. \tag{2.1}$$

There is a partner coordinate b_i only for $i \in M$. More explicitly, the coordinate ring is

$$\mathbb{C}[a_1, \dots, a_d, b_i \ (i \in M), a_f^{-1} \ (f \in F)]/(a_i b_i - P_i - Q_i)_{i \in M}.$$

One may regard this as a closed subvariety of $\mathbb{C}^{d + |M| + |F|}$ by adjoining coordinates for the frozen inverses and the equations $a_f a_f^{-1} = 1$. All frozen coordinates remain variables and are

inverted in the coordinate ring; no fixed fiber of these coordinates is implicit. Write $T \subset Y_B$ for the open set where every a_j is nonzero. Its partner coordinates are uniquely determined, so $T \cong (\mathbb{C}^*)^d$. We will prove that Y_B is smooth of complex dimension d; skew-symmetry and $\det B = 1$ imply that d is even.

The unimodularity condition is imposed on the entire matrix B, including both mutable and frozen indices; it is not merely a rank condition on the mutable submatrix. In particular, every integral skew-symmetric acyclic $r \times r$ exchange matrix B_0 has the principal extension

$$B = \begin{pmatrix} B_0 & I_r \\ -I_r & 0 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 0 & -I_r \\ I_r & B_0 \end{pmatrix}, \qquad \det B = 1.$$

Taking the first block to be mutable and the second block frozen gives a model of complex dimension $2r$. This is the total space with its coefficient variables retained and invertible, rather than a specialized coefficient fiber.

On T the ratio $q_i = P_i/Q_i$ is defined for every mutable i. When the coordinates outside an independent set $I \subset M$ are inverted, Q_i is a unit for $i \in I$, and we use the normalized partner $\widetilde{b}_i = b_i/Q_i$ there. Thus the corresponding exchange equation is $a_i\widetilde{b}_i = 1 + q_i$. No ratio q_j at a boundary point is used unless its denominator is a unit on the chart in question.

Theorem 2.1. *The variety Y_B is smooth. The form*

$$\Omega = \frac{1}{2}\sum_{i,j} B_{ij}\, d\log a_i \wedge d\log a_j$$

on T extends to a holomorphic symplectic form on Y_B. There is an explicit primitive α of $\operatorname{Im}\Omega$ whose Liouville vector field is complete in both directions and of finite type. Its full trapped set is

$$K_B = \{|a_f| = 1 \ (f \in F), \quad |a_i| \le 1, \ (1 - |a_i|^2)b_i = 0 \ (i \in M)\}. \tag{2.2}$$

For each independent subset $I \subset M$, the corresponding closed cap is isomorphic, as a manifold with corners, to

$$(D^2)^{|I|} \times (S^1)^{d-2|I|}.$$

The disk factors fill the original a_i-phase circles. Moreover every smooth Lagrangian component of K_B has a proper exact Lagrangian, cylindrical at infinity, intersecting K_B transversely in one point of that component.

For a primitive α of a symplectic form ω, its Liouville vector field Z is defined by $\iota_Z\omega = \alpha$. Here the trapped set consists of points with relatively compact forward Liouville orbit. Finite type means that the Liouville manifold is the completion of a compact Liouville domain. The theorem does not assert that the primitive is Weinstein or that it agrees with an ordinary affine Kähler primitive.

2.1 Independent zero sets and node charts

Lemma 2.2. *At every point of Y_B, the set $I = \{i \in M : a_i = 0\}$ is independent. For any independent I, the open subset obtained by inverting all a_j, $j \notin I$, is algebraically isomorphic to*

$$\mathcal{N}^{|I|} \times (\mathbb{C}^*)^{d-2|I|}, \qquad \mathcal{N} = \{(u, v) \in \mathbb{C}^2 : uv \neq 1\}. \tag{2.3}$$

These open subsets cover Y_B.

Proof. If the induced quiver on the zero set had an edge, take a source in one of its nontrivial connected components. Exactly one of its two exchange monomials has a zero factor, whereas

the other has no zero factor. This contradicts its exchange equation. Frozen coordinates cannot be zero, so this proves independence.

Fix independent I, and put $L = \{1, \dots, d\} \setminus I$. The rows $B_{I,L}$ form a primitive independent subsystem of $\mathbb{Z}^L$. Indeed the corresponding full rows are part of a basis of $\mathbb{Z}^d$, and their I components vanish. Saturation in the direct summand $\mathbb{Z}^L$ follows from saturation in $\mathbb{Z}^d$. Thus the characters

$$q_i = P_i/Q_i \qquad (i \in I)$$

extend to a Laurent coordinate system in the variables a_j, $j \in L$. On this open set Q_i is a unit for $i \in I$. Setting $\widetilde{b}_i = b_i/Q_i$, the corresponding equations become

$$a_i \widetilde{b}_i = 1 + q_i, \qquad q_i \neq 0.$$

All other exchange partners are eliminated since their a_j are units. Eliminating the q_i gives (2.3). Choosing the actual zero set at a point proves the covering assertion. □

Lemma 2.3. *The logarithmic form Ω extends smoothly and nondegenerately over each chart (2.3).*

Proof. The normal expression is regular:

$$d \log a_i \wedge d \log q_i = \frac{da_i \wedge d\widetilde{b}_i}{a_i \widetilde{b}_i - 1}. \tag{2.4}$$

For completeness, an integral splitting verifies nondegeneracy simultaneously in all zero directions. Set $\Pi = B^{-1}$ and, temporarily on the torus, write $\ell_i = \log a_i$ and $m_i = \log q_i$. With the inverse-matrix convention for the Poisson bracket,

$$\{\ell_i, m_j\} = -\delta_{ij}, \qquad \{m_i, m_j\} = 0 \quad (i, j \in I), \qquad \{\ell_i, \ell_j\} = \Pi_{ij}.$$

Choose an integral matrix C indexed by I with $C_{ij} - C_{ji} = -\Pi_{ij}$. Replace ℓ_i by $\ell_i + \sum_{j \in I} C_{ij} m_j$. The resulting pairs with m_i are mutually orthogonal hyperbolic pairs. Their integral span is a direct symplectic summand: the projection to it is integral because the pairing on each pair is ± 1. The orthogonal complement is again unimodular.

Explicitly, the new normal coordinates are

$$u_i = a_i \prod_{j \in I} q_j^{C_{ij}}, \qquad v_i = \widetilde{b}_i \prod_{j \in I} q_j^{-C_{ij}}; \qquad u_i v_i = 1 + q_i.$$

The complementary characters have zero exponents in the a_i, $i \in I$: orthogonality to every m_i implies precisely this condition. They are therefore units on the chart. All the changes of coordinates extend regularly, and

$$\Omega = \sum_{i \in I} \frac{du_i \wedge dv_i}{u_i v_i - 1} + \Omega_{\mathrm{res}},$$

where Ω_{res} is a nondegenerate logarithmic form on the complementary torus. This proves the assertion. In particular $\omega = \operatorname{Im} \Omega$ is a real symplectic form. □

2.2 The primitive and its extension

On T write

$$A_i = \log |a_i|, \qquad \theta_i = \arg a_i, \qquad S_i = (BA)_i, \qquad T_i = \arg(-q_i).$$

The functions T_i are circle-valued. Fix a small $\varepsilon > 0$ and a smooth function $\chi : \mathbb{R} \to [0,1]$ which equals 1 on $(-\infty, -\varepsilon]$ and 0 on $[-\varepsilon/2, \infty)$. Define

$$\chi_i(A) = \prod_{\substack{j \in M \\ B_{ij} \neq 0}} \chi(A_i - A_j). \tag{2.5}$$

The empty product is 1. For adjacent mutable vertices i, j, the functions χ_i and χ_j cannot both be nonzero.

Choose a compactly supported smooth function $\rho : \mathbb{R} \to [0,1]$ equal to 1 near zero. Choose a smooth periodic real function ϑ supported in a small arc about $0 \in \mathbb{R}/2\pi\mathbb{Z}$, equal to T near zero, and satisfying

$$\vartheta'(T) \leq 1, \qquad T\vartheta(T) \geq 0$$

in the chosen arc. These requirements are compatible: taper the function T to zero on each side using a cutoff with suitably decreasing slope. All its derivatives are bounded.

We shall choose a sufficiently large positive constant κ. Choose a smooth $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{aligned} h \leq 0, \qquad 0 \leq h' \leq 1, \qquad h(A) > A \quad (A < 0), \\ h(A) = 0 \quad (A \geq -2\varepsilon), \qquad h(A) = A + \kappa \quad (A \ll 0). \end{aligned} \tag{2.6}$$

Such a function exists for every sufficiently large κ. For example take h' to decrease smoothly from 1 to 0, integrate with $h = 0$ near -2ε, and adjust the length of the transition interval. This gives any sufficiently large value of the intercept κ.

Define on T

$$H = \sum_{i \in M} h(A_i)\chi_i(A)\rho(S_i)\vartheta(T_i), \qquad \alpha = A^T B\, d\theta - dH. \tag{2.7}$$

The differential satisfies $d\alpha = \omega$ on the torus. The dominance factors are essential for mixed orientations: a ratio such as a_3/a_1 may remain bounded while both numerator and denominator vanish. Consequently a cutoff depending only on S_i would not suppress all neighboring corrections near a zero stratum.

Lemma 2.4. *The form (2.7) extends smoothly to all of Y_B, with $d\alpha = \omega$.*

Proof. Fix a point with independent zero set I. On a sufficiently small neighborhood all neighbors of I are units. For $i \in I$, the dominance factor χ_i is identically one near the zero stratum, and every correction at a mutable neighbor of I is identically zero. Put

$$L_i = \operatorname{Log}(1 - a_i \widetilde{b}_i) = S_i + \mathrm{i}T_i,$$

using the branch near 1. Shrinking the neighborhood, $\rho = 1$ and $\vartheta(T_i) = T_i$, and $h(A_i) = A_i + \kappa$ where $a_i \neq 0$. Because $B_{I,I} = 0$, the potentially singular part of $A^T B\, d\theta$ is the sum of $A_i\, dT_i - S_i\, d\theta_i$. Subtracting the corresponding part of dH gives

$$A_i\, dT_i - S_i\, d\theta_i - d((A_i + \kappa)T_i) = -\operatorname{Im}\left(L_i \frac{da_i}{a_i} + \kappa\, dL_i\right). \tag{2.8}$$

Here $L_i/a_i = -\widetilde{b}_i + O(a_i)$, so the right-hand side is smooth. Corrections at vertices not adjacent to I involve only nonzero coordinates and are already smooth. This handles all zero directions simultaneously. The equality $d\alpha = \omega$ extends from the dense torus by continuity. □

2.3 Completeness and the full trapped set

Let Z be defined by $\iota_Z\omega = \alpha$. Direct contraction gives, on T,

$$\begin{aligned} ZA_i &= A_i - h(A_i)\chi_i(A)\rho(S_i)\vartheta'(T_i) \quad (i \in M), \\ ZA_f &= A_f \qquad\qquad\qquad\qquad\qquad\quad (f \in F), \qquad BZ\theta = \partial_A H. \end{aligned} \tag{2.9}$$

In particular derivatives of χ do not occur in the radial equation. Since $h \le 0$ and $\chi_i\rho\vartheta' \le 1$, this implies

$$A_i > 0 \Longrightarrow ZA_i = A_i > 0, \qquad A_i < 0 \Longrightarrow ZA_i \le A_i - h(A_i) < 0. \tag{2.10}$$

At $A_i = 0$ the derivative vanishes.

Lemma 2.5. *The vector field Z is complete in both directions.*

Proof. All cutoff derivatives are bounded and $|h(A)| \le C(1 + |A|)$. Consequently (2.9) gives

$$|ZA| + |Z\theta| \le C'(1 + |A|)$$

on the torus, with angular components evaluated in local lifts. Gronwall's inequality prevents any logarithmic radius from becoming infinite in finite time. Thus no initially nonzero a_i reaches zero or infinity in finite time, and the exchange equations bound the b_i on every finite time interval. This proves completeness for trajectories starting in T.

One must also consider initial points on the zero strata. Near an independent zero set I, put $w_i = a_i\widetilde{b}_i$. In the neighborhood used in (2.8), the neighboring corrections vanish and

$$ZS_i = S_i, \qquad ZT_i = T_i, \qquad Zw_i = (w_i - 1)\operatorname{Log}(1 - w_i).$$

Moreover $ZA_i = -\kappa$ where $a_i \neq 0$. The angular component can be written as a smooth real function ν_i, so

$$Za_i = (-\kappa + \mathrm{i}\nu_i)a_i.$$

To see that the zero stratum is invariant without dividing by a_i, the equation for Zw_i first implies $Za_i = 0$ at $a_i = 0$, $\widetilde{b}_i \neq 0$, and then at all of $a_i = 0$ by continuity. Since

$$\frac{(w-1)\operatorname{Log}(1-w)}{w} \longrightarrow 1,$$

on the zero stratum one obtains

$$Z\widetilde{b}_i = (1 + \kappa - \mathrm{i}\nu_i)\widetilde{b}_i. \tag{2.11}$$

The remaining nonzero logarithmic coordinates still satisfy a linear growth bound. Indeed the apparently unbounded terms $h(A_i)$ for $i \in I$ disappear from their equations: the relevant derivatives of ρ and χ_i vanish and $T_i = 0$. All surviving terms depend only on nonzero coordinates. Formula (2.11) gives exponential growth of $|\widetilde{b}_i|$, and its normalizing monomial Q_i has at most finite growth on finite intervals. This rules out finite-time escape on each zero stratum and proves completeness everywhere. □

Proposition 2.6. *The full trapped set of Z is exactly K_B.*

Proof. Suppose a forward orbit is relatively compact. The frozen logarithmic radii vanish, by their equation $ZA_f = A_f$. There can be no positive mutable radius. Every initially negative A_i tends to $-\infty$: a finite negative limit would contradict the strict upper bound $ZA_i \le A_i - h(A_i) < 0$. Let I consist of these indices together with the initially zero coordinates.

A convergent subsequence of the orbit has zero set exactly I, so Lemma 2.2 makes I independent. Every neighbor of I therefore has modulus one for the whole orbit.

For an initially nonzero $i \in I$, after sufficiently large positive time its logarithmic radius lies in the region where $\chi_i = 1$ and $h' = 1$. Since all neighbors have modulus one, $S_i = 0$, and

$$ZT_i = \vartheta(T_i).$$

The boundedness of b_i, together with the unit moduli of the factors of Q_i, implies $q_i \to -1$. Thus $T_i \to 0$. Near zero the equation is $T_i' = T_i$, so this convergence is possible only if T_i is identically zero for all sufficiently large times.

This conclusion propagates backward. On the entire orbit the neighboring h terms vanish, and the exact angular equation is

$$ZT_i = \partial_{A_i}\big(h(A_i)\chi_i(A)\big)\vartheta(T_i).$$

Its zero solution is unique on every finite interval. Hence $T_i = 0$ throughout and $b_i = 0$. This use of eventual deep-radius dynamics is necessary; χ_i need not equal one at the beginning of the orbit. For initially zero a_i, (2.11) and the boundedness of the orbit likewise force $\widetilde{b}_i = b_i = 0$. Every trapped point therefore satisfies (2.2).

Conversely, take a point of K_B and let $I = \{i : |a_i| < 1\}$. If its induced quiver had an edge, take a source in a nontrivial component. Its exchange partner is zero. One exchange monomial has modulus one and the other has modulus strictly less than one, so their sum cannot vanish. Thus I is independent. Its neighbors have modulus one, and $q_i = -1$ for each $i \in I$. Whenever $h(A_i) \neq 0$, one has $A_i < -2\varepsilon$ and therefore $\chi_i = 1$. The angular components vanish on this set, the negative radii shrink, and the zero exchange partners remain zero. At a disk center the same conclusion follows from (2.11). This proves invariance of K_B.

Finally, K_B is compact. All a_i are bounded, frozen coordinates and their inverses have modulus one, and $b_i = 0$ whenever $|a_i| < 1$. For $|a_i| = 1$, the exchange equation gives $|b_i| \leq 2$. Thus the closed set K_B is bounded in an affine embedding. Its points are trapped, proving the reverse inclusion. □

The cap description now follows directly. For independent I, impose $b_i = 0$ and $|a_i| \leq 1$ for $i \in I$, and $|a_j| = 1$ for $j \notin I$. The equations for $i \in I$ impose $q_i = -1$ on the latter phase torus. The primitive independent rows $B_{I,L}$ make this a connected torus of dimension $d - 2|I|$. The a_i, $i \in I$, range independently in closed disks; all remaining exchange partners are determined. This gives the stated product and identifies the filled circles. Taking all independent I recovers the full trapped set, including every multiple cap.

2.4 Backward control and finite type

Compactness of the trapped set alone is not sufficient to prove finite type. We give the needed control of the negative flow.

Lemma 2.7. *For every compact subset $P \subset Y_B$, its entire backward saturation $\bigcup_{t\leq 0} \phi_t(P)$ has compact closure. Moreover this negative flow approaches K_B uniformly on P.*

Proof. Compactness and Lemma 2.2 give numbers $\delta_P, \eta_P > 0$ such that on P

$$|a_i| < \delta_P \Longrightarrow |a_j| > \eta_P \quad \text{for every neighbor } j \text{ of } i.$$

Along backward flow, every nonzero modulus moves monotonically toward one by (2.10). All initial upper bounds, frozen inverse bounds, and the indicated neighbor lower bounds therefore persist. Choose $0 < \delta < \min(\delta_P, 1)$ so small that whenever a relevant trajectory has $|a_i| < \delta$, one has $\chi_i = 1$, all neighboring corrections zero, and $h(A_i) = A_i + \kappa$ with $A_i \leq -\kappa$.

In this deep region $\widetilde{b}_i = b_i/Q_i$ is defined and

$$\begin{aligned} ZS_i = S_i, \qquad ZT_i = \rho(S_i)\vartheta(T_i), \\ Z\log|\widetilde{b}_i| = L(S_i,T_i) - ZA_i, \end{aligned} \tag{2.12}$$

where the last equation is used for $\widetilde{b}_i \neq 0$ and

$$L(S,T) = \operatorname{Re}\left[\frac{e^{S+\mathrm{i}T}}{e^{S+\mathrm{i}T}-1}(S+\mathrm{i}\rho(S)\vartheta(T))\right]. \tag{2.13}$$

This function has removable value 1 at $(0,0)$. Near that point the expression inside the real part is $e^z z/(e^z-1)$, with $z = S+\mathrm{i}T$. It has a global lower bound: as $S \to +\infty$ it is asymptotic to S, as $S \to -\infty$ it tends uniformly to zero, and the remaining cylinder is compact after resolving $(0,0)$. Write $L \geq -C$. The constant depends only on ρ, ϑ, not on h or κ.

We now make the earlier choice of κ so that $\kappa > C+1$. In the deep region, setting $c = \rho(S_i)\vartheta'(T_i) \leq 1$,

$$ZA_i = A_i - (A_i+\kappa)c = -\kappa + (A_i+\kappa)(1-c) \leq -\kappa.$$

Consequently $Z\log|\widetilde{b}_i| > 1$. The modulus of $\widetilde{b}_i$ cannot increase under negative flow while in the deep region. Its zero set is invariant, as follows from the exchange equation and (2.12), so the assertion includes $\widetilde{b}_i = 0$. At $a_i = 0$ the same assertion follows from (2.11).

A backward trajectory cannot enter the deep region from outside: if $0 < |a_i| < 1$, its modulus increases backward, and if $|a_i| \geq 1$ it stays at least one. Thus the bound for $\widetilde{b}_i$ starts from the fixed compact initial set. The normalizing monomial Q_i is bounded using the neighbor bounds. Outside the deep region the exchange equation and $|a_i| \geq \delta$ bound b_i. Every affine coordinate and every inverted frozen coordinate is therefore uniformly bounded on the backward saturation of P, which has compact closure Q_P.

For uniform attraction, suppose there are $p_n \in P$, $t_n \to +\infty$ and a neighborhood U of K_B such that $\phi_{-t_n}(p_n) \notin U$. A subsequence converges in Q_P to a point $q \notin U$. For every fixed $s \geq 0$,

$$\phi_s(q) = \lim_{n\to\infty} \phi_{-(t_n-s)}(p_n) \in Q_P.$$

Thus q is trapped and belongs to K_B, a contradiction. □

Proposition 2.8. *The Liouville manifold (Y_B, α) is of finite type.*

Proof. Choose compact neighborhoods $N_0 \subset \operatorname{int} N_1$ of K_B, a smooth proper exhaustion R_0, and a smooth nonnegative function Ψ which vanishes on N_0 and satisfies $\Psi \geq 1+|ZR_0|$ outside N_1. Put

$$F(p) = \int_{-\infty}^{0} \Psi(\phi_t(p))\,dt.$$

Lemma 2.7 makes the integrand identically zero before a common negative time locally in p. Hence F is smooth, and differentiating along the flow gives $ZF = \Psi$.

For $p \notin N_1$, take the last exit time from N_1 along the orbit ending at p. It exists because the negative tail lies in N_0. Integration after that exit gives

$$F(p) \geq R_0(p) - \max_{N_1} R_0.$$

The same lower bound is automatic inside N_1. Thus F is proper. A level $F = c$ with $c > \max_{N_1} F$ lies where $ZF \geq 1$, so it is a smooth compact hypersurface transverse to Z. The domain $\{F \leq c\}$ is a Liouville domain. Every exterior point reaches its boundary in finite negative time. Since $\mathcal{L}_Z\alpha = \alpha$, Liouville flow identifies the exterior with the positive symplectization of the boundary. This proves that its completion is all of Y_B. □

2.5 Actual generalized cocores

We now give the final assertion of Theorem 2.1. This argument uses acyclicity and unimodularity but no bipartite assumption.

Proposition 2.9. *For each smooth Lagrangian component of K_B there is a contractible proper exact Lagrangian, cylindrical at infinity, meeting K_B transversely in exactly one point of that component. The explicit representatives below satisfy $\alpha|_L = 0$.*

Proof. Fix independent $I \subset M$, write $s = |I|$ and $L_0 = \{1, \dots, d\} \setminus I$, and put

$$R_I = M \setminus (I \cup N(I)).$$

The rows of B indexed by $I \cup R_I$ have zero entries in the I columns and form a primitive independent subsystem in $\mathbb{Z}^{L_0}$. Choose phases θ_j, $j \in L_0$, such that

$$\arg q_i = \pi \quad (i \in I), \qquad \arg(-q_j) \notin \operatorname{supp} \vartheta \quad (j \in R_I). \tag{2.14}$$

The associated homomorphism of phase tori is surjective with connected kernel, so these choices exist; one may take $\arg q_j = 0$ for $j \in R_I$.

Define $C_I \subset Y_B$ by

$$\begin{aligned} a_i = 0, \qquad b_i \in \mathbb{C} \quad (i \in I), \\ a_j = e^{A_j + \mathrm{i}\theta_j} \quad (j \in L_0), \qquad B_{I,L_0} A_{L_0} = 0. \end{aligned} \tag{2.15}$$

The other b_j are determined by their exchange equations. The real linear constraints have rank s, and the phase constraints force $q_i = -1$ for $i \in I$, so this is a smooth embedded submanifold with

$$C_I \cong \mathbb{C}^s \times \mathbb{R}^{d-2s}, \qquad \dim_{\mathbb{R}} C_I = d.$$

We first verify strict exactness. In the smooth normal expression (2.8), the pullback of each term to C_I vanishes: a_i and L_i are identically zero there, and the smooth extension of $L_i\, da_i/a_i$ is a multiple of da_i. All remaining angular differentials vanish because the nonzero coordinate phases are fixed. The dominance cutoff kills every correction at a neighbor of I, while (2.14) kills every correction at a vertex of R_I. Hence

$$\alpha|_{C_I} = 0.$$

Its differential vanishes and the dimension is half that of Y_B, so C_I is Lagrangian.

We prove properness without evaluating a monomial ratio at a neighbor of a zero coordinate. Suppose points of C_I converge in Y_B, and let J be the zero set at the limit. Then $I \subset J$ and J is independent. If a new $j \in J \setminus I$ existed, it would have no neighbor in J, and in particular no neighbor in I. Thus $j \in R_I$, and both P_j and Q_j would have finite nonzero limits. Their ratio has the fixed phase in (2.14), different from π, so $P_j + Q_j$ could not vanish. This contradicts the limiting equation $a_j b_j = P_j + Q_j$. No non-I coordinate can therefore vanish in such a limit. All fixed-ray and logarithmic constraints in (2.15) persist, proving that C_I is closed. Its displayed parametrization has continuous inverse, so it is a proper embedding.

Because $\iota_Z \omega = \alpha$ and C_I is Lagrangian with $\alpha|_{C_I} = 0$, the vector field Z is tangent to C_I. Completeness and properness make its full flow preserve C_I. At a sufficiently large contact-type level from Proposition 2.8, Z is transverse to the level and tangent to C_I. Their intersection is therefore Legendrian, and negative flow brings every exterior point of C_I to that intersection. This proves cylindricality.

At a point of $C_I \cap K_B$, all b_i, $i \in I$, vanish. If another mutable coordinate had modulus strictly below one, the independent negative-radius set of K_B would force its index to belong to

R_I. Its fixed phase prevents cancellation in its exchange equation, whereas the K_B condition requires its partner to vanish. This is impossible. Thus there is exactly one intersection, specified by

$$a_I = b_I = 0, \qquad A_{L_0} = 0,$$

with the chosen phases. The phase conditions avoid all other incident cap walls, so this point lies in the smooth locus of the I-cap. The tangent space of K_B consists of its disk-base directions and the residual angular directions; that of C_I consists of the normal fiber directions and the residual logarithmic-radius directions. In the node charts these spaces are complementary. The intersection is transverse.

Finally, the smooth interior of the I-cap is obtained by excluding the remaining walls $q_j = -1$, $j \in R_I$, from its residual phase torus. The character map to these phases has connected kernel, by the primitive subsystem property used above. Each circle with one point deleted is connected, so this interior is connected and contains a point satisfying (2.14). The constructions therefore meet every smooth Lagrangian component. If a specified other smooth point is desired, move the intersection along a path in that component using a compactly supported Hamiltonian cotangent lift in a tubular neighborhood of the path. This preserves the skeleton locally and is the identity near its singular locus and at infinity. It preserves exactness, properness and cylindricality. □

Remark 2.10. For a general acyclic quiver, a vertex neighboring I can have both an incoming and an outgoing neighbor in I. At $a_I = 0$ both its exchange monomials can then vanish. In particular its ratio q_j need not be defined. The properness proof above deliberately uses the independent zero set of a finite limit, and never assumes that one of these neighboring exchange monomials is a unit.

The propositions prove all assertions of Theorem 2.1. The passage from the geometric generalized cocores to generation or to a Fukaya–microsheaf comparison requires the corresponding categorical comparison theorem and its additional ambient hypotheses.

3 Comparison with an affine Kähler form

We prove an exact comparison theorem under two growth estimates. The argument applies to the primitive constructed above once those estimates are verified. It does not require a Weinstein function, or even completeness of the initial Liouville vector field. The conclusion concerns exact symplectomorphisms of open manifolds; it does not assert a deformation through Liouville domains.

3.1 The analytic comparison theorem

Let $X \subset \mathbb{C}^N$ be a smooth closed affine submanifold, with complex structure I. Write $F_1, \dots, F_N$ for its coordinate functions and put

$$R = e + \sum_{\nu=1}^{N} |F_\nu|^2, \qquad \varphi = \frac{R - e}{4}, \qquad \beta = -d\varphi \circ I = \frac{1}{2} \operatorname{Im} \sum_{\nu=1}^{N} \overline{F_\nu}\, dF_\nu. \tag{3.1}$$

Then $\omega_K = d\beta$ is the restriction of the standard Kähler form, and $g(v, w) = \omega_K(v, Iw)$ is the induced Euclidean metric. All norms below are intrinsic norms for the indicated metric. In particular, the norm of an inverse symplectic form is its operator norm from covectors to vectors.

Theorem 3.1. *Suppose that α is a smooth real one-form on X, that $\omega_L = d\alpha$ is symplectic, and that*

$$\omega_L(Iv, Iw) = -\omega_L(v, w).$$

Assume that a constant $C > 0$ satisfies

$$\|\omega_L^{-1}\|_g \le CR, \qquad |\alpha|_g \le C\sqrt{R}(1+\log R) \tag{3.2}$$

throughout X. There are a diffeomorphism $F : X \to X$ isotopic to the identity and a smooth function $f : X \to \mathbb{R}$ such that

$$F^*\omega_K = \omega_L, \qquad F^*\beta = \alpha + df. \tag{3.3}$$

An example of an I-anti-invariant form is $\omega_L = \operatorname{Im}\Omega$, where Ω is a holomorphic symplectic form. The theorem requires a global primitive and a global nondegeneracy statement. Neither follows from the estimates alone.

Lemma 3.2. *Let η be a Kähler form, let $h(v,w) = \eta(v, Iw)$, and define C by $\omega_L(v,w) = h(Cv,w)$. Then*

$$C^* = -C, \qquad CI = -IC,$$

and, for real s, t,

$$(I+sC)^*(I+sC) = \mathrm{id} - s^2C^2, \tag{3.4}$$

$$(C+tI)^*(C+tI) = t^2\,\mathrm{id} - C^2. \tag{3.5}$$

In particular, the relevant operators are invertible and

$$\|(I+sC)^{-1}\|_h \le 1, \qquad \|(C+tI)^{-1}C\|_h \le 1. \tag{3.6}$$

Moreover, CI is skew-adjoint and commutes with C^2.

Proof. Skew-adjointness of C follows from skew-symmetry of ω_L. Anti-invariance gives $CI = -IC$. Since $I^* = -I$, multiplication proves (3.4) and (3.5). The operator $-C^2$ is positive definite, because ω_L is nondegenerate. Finally,

$$\big((C+tI)^{-1}C\big)^*\big((C+tI)^{-1}C\big) = (-C^2)(t^2\,\mathrm{id} - C^2)^{-1} \le \mathrm{id}\,.$$

The remaining assertions follow directly from $CI = -IC$. □

Here and below, I denotes the complex structure and id denotes the identity operator. This distinction is useful in applying the matrix identities.

Lemma 3.3. *Set $L = \log R \ge 1$ and define*

$$\psi = \frac{1}{6}L^{3/2}, \qquad k(R) = \frac{\sqrt{L}}{R}, \qquad \beta_0 = k\beta = -d\psi \circ I, \qquad \gamma = d\beta_0. \tag{3.7}$$

Then γ is Kähler. Its metric h satisfies

$$\frac{1}{2R\sqrt{L}}\, g \le h \le \frac{\sqrt{L}}{R}\, g. \tag{3.8}$$

More generally, let $k_u = (1-u)k + u$, $0 \le u \le 1$. The form $\eta_u = d(k_u\beta)$ is Kähler. If Z_K and Z_u are the Liouville vector fields of β and $k_u\beta$, respectively, then

$$a := Z_K R = \frac{1}{4}|dR|_g^2 \le R - e, \qquad Z_u = \frac{k_u}{k_u + k_u' a}\, Z_K. \tag{3.9}$$

Proof. The ambient radial covector restricts to dR, so

$$|dR|_g \le 2\sqrt{R-e}, \qquad |\beta|_g = \frac{1}{4}|dR|_g \le \frac{1}{2}\sqrt{R-e}. \tag{3.10}$$

Also $Z_K = \nabla_g \varphi$, proving the formula for a.

We have $0 < k \le e^{-1} < 1$, $k' < 0$, and

$$k + Rk' = \frac{1}{2R\sqrt{L}} > 0.$$

Since $\gamma = k\omega_K + k'\, dR \wedge \beta$, its metric is

$$h(v,v) = k|v|_g^2 + \frac{k'}{4}(dR(v)^2 + dR(Iv)^2).$$

The covectors dR and $dR \circ I$ are orthogonal and have equal norm. Consequently

$$dR(v)^2 + dR(Iv)^2 \le 4(R-e)|v|_g^2,$$

which proves (3.8). The same calculation applies to k_u, since $k_u > 0$, $k_u' \le 0$, and $k_u + Rk_u' > 0$.

Finally, $\beta(Z_K) = 0$, and contraction gives

$$\iota_{Z_K} d(k_u \beta) = (k_u + k_u' a)\beta.$$

The coefficient is positive because $a < R$ and $k_u' \le 0$. This proves (3.9). □

Proof of Theorem 3.1. We use three paths of primitives. For a path λ_t with nondegenerate $\omega_t = d\lambda_t$, our Moser vector field satisfies

$$\iota_{X_t}\omega_t = -\dot{\lambda}_t. \tag{3.11}$$

We prove completeness in both parameter directions on each path.

First path. Let

$$\lambda_t = \alpha + t\beta_0, \qquad \omega_t = \omega_L + t\gamma, \qquad 0 \le t \le 1.$$

Use the metric h of Lemma 3.3, and write $\omega_L = h(C\cdot,\cdot)$. All ω_t are nondegenerate by Lemma 3.2. Let W be defined by $\iota_W \omega_L = -\beta_0$. Then

$$X_t = (C + tI)^{-1} CW, \qquad |X_t|_h \le |W|_h.$$

The first estimate in (3.2) gives

$$|W|_g \le CRk|\beta|_g \le C\sqrt{R}\sqrt{L}.$$

By (3.8), changing from h to g loses at most a factor $\sqrt{2L}$. Thus

$$|X_t|_g \le C\sqrt{R}\, L, \qquad |X_t R| \le CR \log R, \tag{3.12}$$

uniformly for $t \in [0,1]$, including $t = 0$. The integral $\int^\infty dr/(r \log r)$ diverges. Hence R stays bounded on every finite parameter interval in either direction. Since R is proper and X_t is smooth, its flow exists for the full interval in both directions.

Second path. Now let

$$\lambda_u = \alpha + k_u\beta, \qquad \omega_u = \omega_L + \eta_u, \qquad 0 \le u \le 1.$$

Write h_u for the metric of η_u, and put

$$\psi_u = (1-u)\psi + u\varphi, \qquad v_u = \nabla_{h_u}\psi_u, \qquad \omega_L = h_u(C_u\cdot,\cdot), \qquad P_u = \mathrm{id} - C_u^2.$$

The Moser equation becomes

$$(C_u + I)X_u = -\frac{1-k}{k_u}\, I v_u.$$

Since $(C_u + I)^2 = -P_u$, this gives

$$X_u = \frac{1-k}{k_u}\, P_u^{-1}(C_u I - \mathrm{id})v_u. \tag{3.13}$$

The operator $P_u^{-1}C_u I$ is skew-adjoint. Therefore

$$d\psi_u(X_u) = -\frac{1-k}{k_u}\, h_u(v_u, P_u^{-1}v_u) \le 0.$$

Use $P_u \ge \mathrm{id}$, $d\psi_u = (k_u/4)dR$, and (3.9). They give

$$0 \le -X_u R \le \frac{(1-k)a}{k_u + k_u' a}. \tag{3.14}$$

In particular, forward trajectories remain in compact sublevel sets of R.

For reverse trajectories, define

$$Q_u(R) = Rk_u(R) = (1-u)\sqrt{\log R} + uR. \tag{3.15}$$

The functions Q_u are uniformly proper, since $Q_u(R) \ge \sqrt{\log R}$. Along a trajectory,

$$\begin{aligned}\frac{d}{du}Q_u(R(u)) &= (1-k)R + (k_u + Rk_u')X_u R \\ &\ge (1-k)\left(R - \frac{a(k_u + Rk_u')}{k_u + k_u' a}\right) \\ &= \frac{(1-k)k_u(R-a)}{k_u + k_u' a} \ge 0.\end{aligned}$$

Thus Q_u is nonincreasing in reverse time. Uniform properness bounds R on every reverse trajectory. This proves completeness in both directions. The argument uses a parameter-dependent proper function; forward monotonicity of R alone would not suffice.

Third path. Finally, consider

$$\lambda_s = \beta + s\alpha, \qquad \omega_s = \omega_K + s\omega_L, \qquad 0 \le s \le 1.$$

We will traverse the resulting map in reverse. In the metric g, Lemma 3.2 and the second estimate in (3.2) give

$$|X_s|_g \le |\alpha|_g \le C\sqrt{R}(1 + \log R), \qquad |X_s R| \le CR(1 + \log R).$$

Again the corresponding Osgood integral diverges. Both time directions are complete.

The three endpoint maps give, successively, exact symplectomorphisms from α to $\alpha + \beta_0$, from $\alpha + \beta_0$ to $\alpha + \beta$, and from $\alpha + \beta$ to β. To check exactness, let Φ_t denote any one of the flows. Cartan's formula and (3.11) imply

$$\frac{d}{dt}\Phi_t^* \lambda_t = d(\Phi_t^*(\lambda_t(X_t))).$$

Integration gives a globally defined smooth exact correction. Composition proves (3.3). Each flow starts at the identity, so F is isotopic to the identity. □

Remark 3.4. The theorem places no support condition on f. In particular, it does not assert that F preserves cylindrical ends for the specified primitives. If α is complete and of finite type, then

$$\beta_{\mathrm{sk}} = (F^{-1})^*\alpha$$

is a complete finite-type primitive of ω_K, and F carries the skeleton of α to that of β_{sk}. A comparison with the skeleton or wrapped category of the specified primitive β requires a separate argument.

3.2 A finite quadratic Poisson embedding

We next verify the growth hypotheses for the acyclic affine models. Let B be an invertible integral skew-symmetric $d \times d$ matrix, with indices partitioned into mutable and frozen sets $M \sqcup F$. Assume that the mutable quiver is acyclic. Define

$$P_i = \prod_{B_{ij}>0} a_j^{B_{ij}}, \qquad Q_i = \prod_{B_{ij}<0} a_j^{-B_{ij}},$$
$$Y_B = \{a_i b_i = P_i + Q_i \ (i \in M), \quad a_f \neq 0 \ (f \in F)\}.$$

The original affine generators are the a_j, the b_i, and the a_f^{-1}. We assume in this subsection that Y_B is smooth and that

$$\Omega = \frac{1}{2}\sum_{i,j} B_{ij}\, d\log a_i \wedge d\log a_j$$

extends from the initial torus to a holomorphic symplectic form on Y_B. These are geometric inputs. The estimates below do not replace the proof of those inputs.

Proposition 3.5. *There is a closed affine embedding of Y_B with coordinates $F_1, \dots, F_N$ whose pairwise Poisson brackets are polynomials of degree at most two in the F_ν. Moreover, any prescribed finite set of regular functions can be included among the coordinates of such an embedding. For its induced metric,*

$$\|(\operatorname{Im}\Omega)^{-1}\|_g \leq CR. \tag{3.16}$$

Proof. Write $\Pi = B^{-1}$ and use the bracket convention $\{a_i, a_j\} = \Pi_{ij} a_i a_j$. Changing the overall sign of the bracket would not affect the argument. Let p_i, n_i be the exponent vectors of P_i, Q_i. For a Laurent monomial $A = a^u$ whose mutable exponents are nonnegative, direct differentiation gives

$$\{A, b_j\} = [u^T \Pi(n_j - e_j)]\, A b_j - u_j a^{u - e_j + p_j}. \tag{3.17}$$

Indeed, $p_j - n_j$ is the transpose of the jth row of B, so $\Pi(p_j - n_j) = -e_j$. The second term of (3.17) occurs only when $u_j > 0$ and is regular. For $i \neq j$, the same calculation gives

$$\begin{aligned} \{b_i, b_j\} &= c_{ij} b_i b_j + \frac{[B_{ij}]_+ P_i Q_j - [-B_{ij}]_+ Q_i P_j}{a_i a_j}, \\ c_{ij} &= (n_i - e_i)^T \Pi (n_j - e_j), \end{aligned} \tag{3.18}$$

where $[m]_+ = \max(m, 0)$. Every additional monomial is regular. For example, if $B_{ij} = m > 0$, then $P_i Q_j$ contains $a_i^m a_j^m$.

Choose positive integer weights w_j, $j \in M$, such that

$$w_j > \sum_{\ell \in M} [B_{j\ell}]_+ w_\ell. \tag{3.19}$$

Such weights exist by recursion in reverse topological order. The operation

$$u \longmapsto u - e_j + p_j \qquad (u_j > 0)$$

strictly decreases the nonnegative integer $\sum_{\ell \in M} w_\ell u_\ell$. Start with the monomials a_j, a_f^{-1}, and those appearing in the extra terms of (3.18), and close this finite set under the above operations. Every chain has bounded length and finite branching. Although frozen exponents may change, they change by one of finitely many fixed vectors at each step. Thus the closure $\mathcal{A}$ is finite.

Use the elements of $\mathcal{A}$ and the b_i as coordinates. They contain the original affine generators, so the resulting map is a closed embedding: it is the graph of finitely many regular functions over the original closed embedding. Brackets between two monomials are constant multiples of their products. Equation (3.17) handles brackets with one b_j, and (3.18) handles brackets between two b_i. All coordinate brackets are therefore quadratic.

Let $G_1, \dots, G_m$ be additional regular functions. Express them as polynomials of degree at most s in the coordinates just obtained. Adjoin every monomial of degree at most s, together with the G_j. The Leibniz rule shows that the bracket of two polynomials of degree at most s has degree at most $2s$. Every monomial of degree at most $2s$ is a product of two monomials of degree at most s. Hence all brackets remain quadratic in the enlarged coordinate set. This proves the assertion about prescribed regular functions.

The components of the pushforward Poisson bivector in the ambient coordinates are its coordinate brackets. The quadratic bound gives an ambient bivector norm at most $C(1 + \sum |F_\nu|^2)$. The inclusion is isometric on tangent spaces, so the same bound holds intrinsically. The inverse of $\operatorname{Im}\Omega$ is obtained from the holomorphic Poisson tensor by a fixed real-linear operation. This proves (3.16), with a possibly different constant. □

3.3 The estimate for the primitive

We give the estimate for a primitive of the following form on the initial torus. Put

$$A_i = \log|a_i|, \qquad \theta_i = \arg a_i, \qquad S_i = \log|P_i/Q_i|, \qquad T_i = \arg(-P_i/Q_i),$$

and let

$$\alpha = A^T B\, d\theta - dH, \qquad H = \sum_{i \in M} h(A_i)\chi_i(A)\rho(S_i)\vartheta(T_i). \tag{3.20}$$

Here h has bounded derivative, grows at most linearly, vanishes for A sufficiently large, and equals $A + \kappa$ for A sufficiently negative. The function ρ is smooth and compactly supported, with $\rho = 1$ near zero. The function ϑ is a smooth circle function which equals the local angular coordinate near zero. Finally,

$$\chi_i(A) = \prod_{\substack{j \in M \\ B_{ij} \neq 0}} \chi(A_i - A_j),$$

where χ is smooth, equals one for $s \leq -\varepsilon$, and equals zero for $s \geq -\varepsilon/2$. These assumptions suffice for the estimate; the additional inequalities used to determine the Liouville dynamics are not needed here.

Proposition 3.6. *Suppose that* (3.20) *extends to a smooth primitive of* $\operatorname{Im}\Omega$ *on* Y_B. *The embedding in Proposition 3.5 can be chosen so that*

$$|\alpha|_g \leq C\sqrt{R}(1 + \log R). \tag{3.21}$$

Thus Theorem 3.1 applies to this primitive.

Proof. We first record the regular functions needed near a zero coordinate. At any point of Y_B, the mutable zero set is independent. Otherwise, a nontrivial connected component of its induced acyclic quiver has a source i. In the exchange equation for i, one monomial contains a zero outgoing neighbor, whereas the other has only nonzero factors. This contradicts $a_i b_i = 0$. Thus adjacent coordinates have no common zero, and Q_i has no zero on $\{a_i = 0\}$. The unit-ideal criterion provides regular functions such that

$$a_j c_{ij} = 1 + a_i d_{ij} \quad (j \sim i), \qquad Q_i c_i = 1 + a_i d_i, \qquad g_i = b_i c_i. \tag{3.22}$$

Include all these functions among the embedding coordinates using Proposition 3.5. Their values are bounded by $\sqrt{R}$, and their differentials have norm at most one.

Fix a sufficiently small constant $\delta > 0$. At a point of Y_B, put

$$J = \{i \in M : |a_i| < \delta/\sqrt{R}\}.$$

This set is used only for a pointwise estimate; we never differentiate its dependence on the point. If $i \in J$ and $j \sim i$, then (3.22) gives

$$|a_j| \geq \frac{1-\delta}{\sqrt{R}}.$$

For δ small enough, all the factors defining χ_i are on their constant-one region, and all neighboring mutable corrections are identically zero in a neighborhood of the point. In particular, J is independent.

Set $\widetilde{b}_i = b_i/Q_i$. The exchange equation gives $-P_i/Q_i = 1 - a_i\widetilde{b}_i$. For $i \in J$, choose the logarithm near one and write

$$L_i = \operatorname{Log}(1 - a_i\widetilde{b}_i) = \operatorname{Log}(1 + a_i(d_i - g_i)) - \operatorname{Log}(1 + a_i d_i). \tag{3.23}$$

Both arguments lie in a fixed small disk around one. The cutoff factors ρ and ϑ consequently agree with their prescribed local expressions. Also $h(A_i) = A_i + \kappa$, after decreasing δ if necessary. The terms in (3.20) involving the ith logarithmic coordinate combine into the regular expression

$$-\operatorname{Im}\left(L_i \frac{da_i}{a_i} + \kappa\, dL_i\right). \tag{3.24}$$

Indeed, $L_i = S_i + \mathrm{i}T_i$, and the corresponding part of the uncorrected primitive is $A_i\, dT_i - S_i\, d\theta_i$. Subtracting $d((A_i + \kappa)T_i)$ gives (3.24). Independent indices permit these cancellations simultaneously.

Formula (3.23), together with the coordinate and differential bounds, implies

$$|L_i/a_i| \leq C\sqrt{R}, \qquad |dL_i|_g \leq C\sqrt{R}.$$

For the first estimate, use $\operatorname{Log}(1 + az)/a = z\int_0^1 (1 + taz)^{-1}\, dt$ on the fixed small disk. The second follows by differentiating the two regular logarithms in (3.23). Thus each expression (3.24) has norm at most $C\sqrt{R}$, including at $a_i = 0$.

For every remaining mutable coordinate, $|a_j| \geq \delta/\sqrt{R}$. For a frozen coordinate, its inverse is an embedding coordinate, so $|a_f| \geq R^{-1/2}$. Consequently all remaining logarithmic coordinates satisfy

$$|\log|a_j|| \leq C(1 + \log R), \qquad |da_j/a_j|_g \leq C\sqrt{R}.$$

Every correction not already treated involves only these coordinates: a correction adjacent to J vanishes with all its derivatives. The cutoff functions have bounded derivatives, and h has at most linear growth. The remaining terms in $A^T B\, d\theta$ and dH are therefore bounded by $C\sqrt{R}(1 + \log R)$. There are finitely many such terms, which proves (3.21). □

Remark 3.7. The preceding propositions verify both estimates in (3.2) using one embedding. They do not infer quadratic Poisson growth from an unspecified polynomial bound: the finite closure and its degree enlargement are part of the proof. Likewise, the bound for α is obtained after cancellation of its apparent logarithmic singularities. Estimating $A^T B\, d\theta$ and dH separately near $a_i = 0$ would not establish it.

3.4 The affine endpoint and its independence of the embedding

We record the end control needed for the affine endpoint. This argument uses algebraicity and compactly supported perturbations. It makes no assertion about a general path between exact primitives.

Lemma 3.8. *Let $X \subset \mathbb{C}^N$ be a smooth closed algebraic submanifold. With*

$$\varphi = \frac{1}{4}\sum_{\nu} |F_\nu|^2, \qquad \beta = -d\varphi \circ I,$$

the Liouville manifold (X, β) is complete and of finite type. It admits a compactly supported deformation through strictly plurisubharmonic potentials to a finite-type Weinstein structure. The endpoint comparison can be chosen to be a compactly supported diffeomorphism G satisfying

$$G^*\beta_1 = \beta + df, \qquad f \in C_c^\infty(X).$$

Proof. The function φ is proper, nonnegative, and strictly plurisubharmonic. Its critical locus is semialgebraic and has finitely many connected components. Each component is joined by piecewise C^1 semialgebraic paths, and $d\varphi = 0$ along such paths. Hence φ is constant on each component and has finitely many critical values. Properness then implies that the entire critical locus is compact.

For the induced Euclidean metric g, the Liouville vector field is $Z = \nabla_g \varphi$, and orthogonal projection of the ambient gradient gives

$$|Z|_g^2 \le \varphi, \qquad Z\varphi = |d\varphi|_g^2 \le \varphi.$$

Thus φ grows at most exponentially in forward time and does not increase in reverse time. Properness proves completeness in both directions. Above the largest critical value, $Z\varphi > 0$, so sufficiently large sublevels are Liouville domains. Every point outside such a domain meets its boundary in finite reverse time: on the intervening compact annulus, $Z\varphi$ has a positive minimum. Forward trajectories leave every compact subset. The Liouville flow therefore identifies X with the completion of the domain.

Choose a compact neighborhood of the critical locus and a larger compact sublevel whose boundary has a collar free of critical points. There is an arbitrarily small smooth perturbation v, supported in the interior of that sublevel, such that $\varphi_1 = \varphi + v$ is Morse. One can obtain it by multiplying the real and imaginary coordinate functions by a cutoff which equals one near the critical locus. Their differentials span the cotangent space there; a generic sufficiently small linear combination makes $d\varphi_1$ transverse to the zero section. Smallness prevents new critical points on the compact transition region.

Strict plurisubharmonicity is open in the C^2 topology on this compact support. We may therefore arrange that every $\varphi_t = \varphi + tv$, $0 \le t \le 1$, is strictly plurisubharmonic. The family is unchanged on the end and near the boundary of the chosen large sublevel. Its Liouville vector fields are complete, and that one compact domain is of contact type for the whole family. At the endpoint, $Z_1 = \nabla_{g_1}\varphi_1$ is gradient-like for the exhausting Morse function φ_1. Its critical points form a finite set, since they are nondegenerate and contained in a fixed compact set. Thus the endpoint is Weinstein of finite type.

Finally, put $\beta_t = -d\varphi_t \circ I$ and solve

$$\iota_{V_t} d\beta_t = -\dot{\beta}_t.$$

The vector fields V_t have support in one compact set. Their flow is complete, equals the identity off a compact set, and satisfies

$$G_t^* \beta_t = \beta + d \int_0^t G_s^*(\beta_s(V_s))\, ds.$$

The correction is compactly supported. Taking $t = 1$ proves the final assertion. □

Remark 3.9. A generic translated squared-distance function also gives a finite-type Weinstein structure. The compact perturbation in Lemma 3.8 has the additional feature that the specified primitive remains unchanged on the end. This feature is useful when transporting boundary data.

Lemma 3.10. *Let $F : X \hookrightarrow \mathbb{C}^N$ and $H : X \hookrightarrow \mathbb{C}^M$ be closed algebraic embeddings of the same smooth affine variety. The primitives obtained by restricting the standard Euclidean primitive through F and H are deformation equivalent as completions of Liouville domains.*

Proof. Write

$$\varphi_0 = \frac{1}{4}|F|^2, \qquad \varphi_1 = \frac{1}{4}|H|^2, \qquad \varphi_t = (1-t)\varphi_0 + t\varphi_1, \qquad \beta_t = -d\varphi_t \circ I.$$

Each φ_t is strictly plurisubharmonic and is the squared norm for the closed embedding $(\sqrt{1-t}\, F, \sqrt{t}\, H)$. Consequently its Liouville field is complete. The family is uniformly proper, since

$$\varphi_t \geq \min(\varphi_0, \varphi_1).$$

We show that its critical points lie in a fixed compact subset of X.

The total critical locus

$$\mathcal{C} = \{(x,t) \in X \times [0,1] : d_x \varphi_t = 0\}$$

is semialgebraic. If its projection to X were unbounded, semialgebraic curve selection at infinity would give a C^1 semialgebraic curve $(x(s), t(s)) \in \mathcal{C}$, $0 < s < \varepsilon$, with $x(s)$ leaving every compact subset as $s \to 0$. Since both endpoint potentials are proper and bounded below,

$$\varphi_0(x(s)) \longrightarrow +\infty, \qquad \varphi_1(x(s)) \longrightarrow +\infty.$$

A one-variable semialgebraic function with this limit has strictly negative derivative for all sufficiently small positive s; this follows from semialgebraic monotonicity, or its leading Puiseux term. But evaluation of $d_x \varphi_{t(s)} = 0$ on $x'(s)$ gives

$$0 = (1 - t(s)) \frac{d}{ds} \varphi_0(x(s)) + t(s) \frac{d}{ds} \varphi_1(x(s)),$$

which is impossible. This proves the uniform compactness assertion.

The critical values are now uniformly bounded. Choose c larger than all of them and put $D_t = \{\varphi_t \leq c\}$. Uniform properness makes $\bigcup_t D_t \times \{t\}$ compact, and each boundary is smooth and of contact type. These domains form a smooth family. To identify them with one fixed domain, choose a vector field along the boundaries satisfying

$$d\varphi_t(W_t) = -\partial_t \varphi_t,$$

and extend it over a common compact neighborhood. Its flow trivializes the family of domains. Pulling back β_t gives a deformation of Liouville structures on a fixed compact domain. The Liouville flow identifies each original affine manifold with the corresponding completion, proving the assertion. □

Remark 3.11. The preceding lemma establishes end control for two algebraic squared-norm potentials. Its proof does not apply to an arbitrary exact change of primitive. In particular, it does not remove the support issue for the correction in Theorem 3.1.

4 Exact transport of wrapped categories

Throughout this section, write

$$\mathcal{C}(D,\lambda) := \operatorname{Perf}\mathcal{W}(D,d\lambda)$$

for the wrapped Fukaya category of the completion of a Liouville domain, after adjoining twisted complexes and taking the idempotent completion. We use the same notation for a completed Liouville manifold. Coefficients and brane data are fixed. All embeddings and deformation identifications below preserve those data; in particular, the graded statement assumes compatible gradings.

A Liouville manifold is *weakly Weinstein* if its core is mostly Lagrangian in the sense of [3, Theorem 1.14], and each component of its smooth Lagrangian part admits a generalized cocore. Here a generalized cocore is a properly embedded exact cylindrical Lagrangian meeting the entire core at exactly one point, transversely at a smooth Lagrangian point. This is a hypothesis in addition to finite type.

Theorem 4.1. *Let (M,λ_0) and (N,λ_1) be complete finite-type Liouville manifolds, each weakly Weinstein. Suppose there is a symplectomorphism $F : M \longrightarrow N$ and a globally defined smooth function $f : M \to \mathbb{R}$ such that*

$$F^*\lambda_1 = \lambda_0 + df. \tag{4.1}$$

Assume that the brane data correspond under F. Then

$$\mathcal{C}(M,\lambda_0) \simeq \mathcal{C}(N,\lambda_1). \tag{4.2}$$

No boundedness or support condition on f is required.

4.1 Restriction functors in a common ambient sector

Let $D_0 \subset \operatorname{int}(D_1)$ be a strict inclusion of Liouville domains: the primitive on D_0 is the restriction of the primitive on D_1. Write $X_1 = \widehat{D}_1$ and use the universal ambient sector

$$V = X_1 \times \mathbb{C}_{\mathrm{Re}\geq 0}.$$

The copies of D_0 and D_1 at the positive boundary of the second factor are Liouville hypersurfaces, hence ribbon stops. Denote their cores by $\mathfrak{c}_0$ and $\mathfrak{c}_1$. Replacing a ribbon stop by its core does not change the partially wrapped category [3, Corollary 3.9]. For $i = 0, 1$, stabilization followed by the tautological stopped inclusion gives an Orlov functor

$$E_i : \mathcal{C}(D_i) \longrightarrow \mathcal{B}_i := \operatorname{Perf}\mathcal{W}(V,\mathfrak{c}_i). \tag{4.3}$$

These functors are fully faithful: stabilization is fully faithful, and the relevant inclusion is forward stopped [3, §11.1, equations (11.2)–(11.4)]. If E_0 is essentially surjective, define restriction by

$$\begin{array}{ccc} \mathcal{C}(D_1) & \xrightarrow{E_1} & \mathcal{B}_1 \\ {\scriptstyle R_{10}}\downarrow & & \downarrow{\scriptstyle S_{10}} \\ \mathcal{C}(D_0) & \xrightarrow{E_0} & \mathcal{B}_0 \end{array} \qquad R_{10} = E_0^{-1} S_{10} E_1, \tag{4.4}$$

where S_{10} is stop removal. Equivalently, one can use the two-stop Viterbo sector of [11, §2.5]; its fixed grading shift is included in the choice of the Orlov functors. We fix this convention throughout. The required essential surjectivity, as in [3, Remark 11.1], follows here from the next lemma.

Lemma 4.2. *If $\widehat{D}_0$ is weakly Weinstein, then the inner Orlov functor E_0 in (4.3) is an equivalence. The same conclusion holds when D_0 is connected to such a domain by a compact Liouville deformation.*

Proof. The core $\mathfrak{c}_0$, viewed as a stop, is compact and mostly Legendrian. By [3, Theorem 1.16], the category $\mathcal{W}(X_1 \times \mathbb{C}_{\mathrm{Re}\geq 0}, \mathfrak{c}_0)$ is generated by the linking disks at its smooth Legendrian points. This theorem does not require X_1 to be Weinstein. Stabilizing a generalized cocore of D_0 and applying the stopped inclusion produces the corresponding linking disk. This is the local identification used in the proof of [3, Theorem 1.14] and in the definition of a homological cocore in [4, §7.1]. Thus every linking-disk generator lies in the essential image of E_0. Since E_0 is fully faithful and its source is idempotent complete and pretriangulated, it is an equivalence.

For the deformation assertion, use the ribbon stop. A compact Liouville deformation gives a smooth family of fibered stops and their associated sectors. By [11, Lemma 2.15], Orlov functors commute with the induced deformation equivalences. Essential surjectivity therefore transports along the family. This argument does not require the core of every intermediate primitive to be mostly Lagrangian. See also [11, Example 2.17], where the transverse-cocore hypothesis gives stop removal. □

If the ribbon D_1 is embedded in the contact boundary of another Liouville sector, the functor R_{10} makes the analogous Orlov square commute: the construction in [3, §11.1] pushes the universal square into the chosen ambient sector. This common-ambient compatibility gives composition.

Lemma 4.3. *Let $D_0 \subset \mathrm{int}(D_1) \subset \mathrm{int}(D_2)$ be strict Liouville subdomains for one primitive. Suppose that the inner Orlov functors needed to define their restrictions are equivalences. Then*

$$R_{10} \circ R_{21} \simeq R_{20}. \tag{4.5}$$

This identification is natural and compatible with further nested inclusions.

Proof. Put all three ribbons in $V = \widehat{D}_2 \times \mathbb{C}_{\mathrm{Re}\geq 0}$ and form $\mathcal{B}_i = \mathrm{Perf}\,\mathcal{W}(V, \mathfrak{c}_i)$ and their Orlov functors E_i. Since the domains use the same primitive, $\mathfrak{c}_0 \subset \mathfrak{c}_1 \subset \mathfrak{c}_2$: an orbit that stays in a smaller domain also stays in a larger one. The stop-removal functors $S_{ji} : \mathcal{B}_j \to \mathcal{B}_i$ satisfy

$$S_{10}S_{21} \simeq S_{20},$$

because they are induced by the corresponding inclusions of stops. The common-ambient version of (4.4) gives

$$E_0R_{10}R_{21} \simeq S_{10}E_1R_{21} \simeq S_{10}S_{21}E_2 \simeq S_{20}E_2 \simeq E_0R_{20}.$$

Full faithfulness of E_0 proves (4.5) as an identification of functors. For a longer finite chain, use a single largest ambient sector. The compatibility is then that of the inclusion functors between the stopped categories. □

Lemma 4.4. *Let $A \subset \mathrm{int}(B)$ be compact manifolds with boundary, and let λ_s, $s \in [0,1]$, be a smooth family making them strict Liouville domains. Assume that the family is constant on fixed neighborhoods of ∂A and ∂B, and that the required inner Orlov functors are equivalences. The deformation equivalences identify the restriction functors at $s = 0$ and $s = 1$.*

Proof. Use the ribbon A inside the universal sector associated to B. The family gives a family of fibered stops and of their stopped sectors. Since it is fixed near both boundaries, their completions form a Liouville deformation with fixed end data. Moser and Gray identifications on the compact domains place this family in a fixed ambient realization. The deformation equivalence can also be implemented by a finite zigzag of trivial sector inclusions.

By [11, Lemma 2.15], the Orlov functors commute with these identifications. The stop-removal arrows do as well: at each stage of the zigzag they are the pushforward functors for the same inclusions with the stop forgotten. Thus the squares (4.4) at the two endpoints are identified. Inverting the inner Orlov equivalence proves the assertion. In particular, no regularity assertion about intermediate cores is needed. □

If $A \subset B$ are radial truncations of the same complete Liouville manifold and both contain its full compact core, restriction is an equivalence. Their completions are identified by their common cylindrical end. In the weakly Weinstein case this also follows from (4.4): the two cores in the common ambient sector agree, stop removal is the identity, and both Orlov functors are equivalences. Call such a restriction a *collar restriction*.

4.2 Four nested domains

Proof of Theorem 4.1. Pull the target structure back by F. On one manifold M we then have

$$\alpha = \lambda_0, \qquad \beta = F^*\lambda_1, \qquad \beta - \alpha = df, \qquad d\alpha = d\beta = \omega. \tag{4.6}$$

This pullback identifies Liouville manifolds with their respective primitives.

Choose an α-radial truncation A_0 containing the full α-core. Choose a β-radial truncation B_0 containing both A_0 and the full β-core in its interior. Continue alternately to obtain

$$A_0 \Subset B_0 \Subset A_1 \Subset B_1. \tag{4.7}$$

Here A_0, A_1 are truncations for α, and B_0, B_1 are truncations for β. Finite type supplies the two radial exhaustions, and compactness allows each successive choice.

The four boundaries are disjoint. Choose a smooth compactly supported function $\chi : M \to [0,1]$ such that

$$\chi = 0 \text{ near } \partial A_0 \cup \partial A_1, \qquad \chi = 1 \text{ near } \partial B_0 \cup \partial B_1,$$

and put

$$\lambda_* := \alpha + d(\chi f). \tag{4.8}$$

Then

$$d\lambda_* = \omega, \qquad \lambda_* = \begin{cases} \alpha & \text{near } \partial A_0 \cup \partial A_1, \\ \beta & \text{near } \partial B_0 \cup \partial B_1. \end{cases}$$

All four domains in (4.7) are strict Liouville subdomains for the single primitive λ_*. The outward-pointing condition follows because the primitive agrees exactly with the original primitive near each boundary.

On either A_i, use the family

$$\alpha + s\, d(\chi f), \qquad 0 \le s \le 1. \tag{4.9}$$

It is constant near ∂A_i and has symplectic form ω. On either B_i, use

$$\beta + s\, d((\chi - 1)f), \qquad 0 \le s \le 1. \tag{4.10}$$

It is constant near ∂B_i and also has symplectic form ω. Both families end at λ_*. These compact Liouville-domain deformations give equivalences

$$\mathcal{C}(A_i, \lambda_*) \simeq \mathcal{C}(M, \alpha), \qquad \mathcal{C}(B_i, \lambda_*) \simeq \mathcal{C}(M, \beta). \tag{4.11}$$

By the deformation assertion of Lemma 4.2, the required inner Orlov functors are equivalences. The three successive inclusions give

$$\mathcal{C}(B_1, \lambda_*) \xrightarrow{h} \mathcal{C}(A_1, \lambda_*) \xrightarrow{g} \mathcal{C}(B_0, \lambda_*) \xrightarrow{r} \mathcal{C}(A_0, \lambda_*). \tag{4.12}$$

Lemma 4.3 identifies rg with restriction for $A_0 \subset A_1$ and gh with restriction for $B_0 \subset B_1$.

Apply (4.9) to the whole pair $A_0 \subset A_1$. The family is fixed near both boundaries, so Lemma 4.4 identifies rg with restriction between the original α-truncations. This is a collar restriction and hence an equivalence. Similarly, (4.10) on the pair $B_0 \subset B_1$ identifies gh with a β-collar restriction, which is also an equivalence.

The functors

$$\ell = (rg)^{-1}r, \qquad q = h(gh)^{-1}$$

are a left inverse and a right inverse of g, respectively. Since

$$\ell \simeq \ell g q \simeq q,$$

they give an inverse of g. Thus g is an equivalence. Equation (4.11) and the initial pullback by F now give (4.2). □

Remark 4.5. The proof uses f only on finitely many compact domains. It neither requires uniformly controlled ends for the family $\alpha + sdf$ on the noncompact manifold nor asserts that F takes cylindrical objects to cylindrical objects. The primitive λ_* is auxiliary to the four chosen domains. The conclusion is an equivalence of categories; it does not assert an ambient isotopy between the two cores.

Remark 4.6. The transport argument uses only wrapped Fukaya categories. The adapted endpoint is computed independently by microlocal gluing and the direct GPS application in Section 7, where the analytic hypotheses of [4, Theorem 1.4] are verified for the actual cutoff primitive. After that computation and the geometric hypotheses of Theorem 4.1 are established, the theorem compares this endpoint with the ordinary affine endpoint.

4.3 Affine comparison

Corollary 4.7. *Let Y_B satisfy the hypotheses of Section 2, and let α be its explicit finite-type Liouville primitive. For a suitable closed algebraic embedding $Y_B \hookrightarrow \mathbb{C}^N$, let β denote the restricted standard Kähler primitive. There are a diffeomorphism $\Phi : Y_B \to Y_B$ isotopic to the identity and a smooth real function f such that*

$$\Phi^*\beta = \alpha + df.$$

For the affine Kähler primitive β' obtained from any closed algebraic embedding of Y_B, there is an equivalence

$$\operatorname{Perf} \mathcal{W}(Y_B, \alpha) \simeq \operatorname{Perf} \mathcal{W}(Y_B, \beta')$$

with compatible brane data.

Proof. Apply Theorem 2.1 and the growth estimates of Section 3. The complete Moser map gives the exact symplectomorphism, and the explicit cocores make the adapted endpoint weakly Weinstein. The affine endpoint can first be deformed on a compact set to the Weinstein endpoint of Lemma 3.8. Its compactly supported Moser map composes with the constructed exact map. Theorem 4.1 applies to the adapted endpoint and this Weinstein endpoint. Compact Liouville deformation then returns to the original affine primitive. Finally, Lemma 3.10 allows any other closed algebraic embedding of the same variety. This argument uses no global wrapped-to-microlocal comparison. □

5 The marked symplectic germ of the cap support

Let $V = M \sqcup F$, let $d = |V|$, and let $B \in \mathrm{GL}_d(\mathbb{Z})$ be skew-symmetric. Its mutable directed graph is acyclic. For $i \in M$, put

$$P_i(a) = \prod_{B_{iv}>0} a_v^{B_{iv}}, \qquad Q_i(a) = \prod_{B_{iv}<0} a_v^{-B_{iv}}, \qquad q_i = P_i/Q_i,$$

and write

$$Y = \{a_i b_i = P_i(a) + Q_i(a) \ (i \in M), \quad a_f \neq 0 \ (f \in F)\}. \tag{5.1}$$

The coordinates a_i, b_i for $i \in M$ are affine coordinates, and a_f for $f \in F$ are invertible coordinates. The rational character q_i will only be used where its denominator is a unit. The compact cap support is

$$K = \{|a_f| = 1 \ (f \in F), \quad |a_i| \leq 1, \quad (1 - |a_i|^2) b_i = 0 \ (i \in M)\} \subset Y. \tag{5.2}$$

On the seed torus put $A_v = \log |a_v|$ and $\theta_v = \arg a_v \in \mathbb{R}/2\pi\mathbb{Z}$. The forms are

$$\Omega = \frac{1}{2} \sum_{v,w} B_{vw} \, d\log a_v \wedge d\log a_w, \qquad \omega = \operatorname{Im} \Omega.$$

An independent set is a subset of mutable vertices containing no pair joined by an edge.

Lemma 5.1. *The zero set of the a-coordinates at a point of Y is independent. At a point of K, the set $\{i : |a_i| < 1\}$ is also independent. For every independent $C \subset M$, the characters q_c, $c \in C$, form a primitive independent subsystem in the character lattice on $V \setminus C$.*

Proof. If a zero set contained an edge, choose a source in a nontrivial component of its induced acyclic graph. In that vertex's exchange equation one monomial is zero and the other is nonzero, a contradiction.

For the second assertion choose a source in a nontrivial component of the induced graph on $\{i : |a_i| < 1\}$. Its exchange partner is zero. One exchange monomial has modulus one, whereas the other has modulus strictly less than one, so they cannot sum to zero. This argument also applies when some coordinates vanish.

The indicated rows of B are part of an integral basis and have zero entries in the C-columns. They therefore form a primitive direct summand already in the complementary coordinate lattice. □

On $U_C = \{a_l \neq 0 : l \notin C\}$, each Q_c is a unit. Put $b'_c = b_c/Q_c$; then $a_c b'_c = 1 + q_c$. Completing the characters of Lemma 5.1 to an integral basis gives

$$U_C \cong \{ab = 1 + q, \ q \neq 0\}^{|C|} \times (\mathbb{C}^*)^{d-2|C|}. \tag{5.3}$$

The charts cover Y, so Y is smooth of dimension d. Moreover,

$$d\log a_c \wedge d\log q_c = \frac{da_c \wedge db'_c}{a_c b'_c - 1}$$

extends across $a_c = 0$. The simultaneous coordinates below show that the extension of Ω is nondegenerate.

The germ of the primitive near K is

$$\lambda = A^T B \, d\theta - dH, \qquad H = \sum_{i \in M} h(A_i) \rho(\log |q_i|) \beta(\arg(-q_i)). \tag{5.4}$$

Here $h = 0$ near $A \geq 0$, $h(A) = A + \kappa$ sufficiently far to the left, $\rho = 1$ near zero, and β is a smooth periodic function equal to the selected argument near zero. The local logarithm

$$\mathcal{Q}_i = \operatorname{Log}(1 - a_i b_i') = \log|q_i| + i \arg(-q_i), \qquad \mathcal{Q}_i/a_i = -b_i' + O(a_i)$$

shows that the primitive extends smoothly. For example, the singular-looking contribution in the linear end of h is

$$-\operatorname{Im}\left(\mathcal{Q}_i \frac{da_i}{a_i} + \kappa\, d\mathcal{Q}_i\right).$$

Terms associated to adjacent vertices vanish near a zero divisor. The primitive restricts to zero on every smooth stratum of K.

The dominance cutoffs in the global primitive of Section 2 equal one wherever a summand can be active near K. On K, a nonzero $h(A_i)$ requires $A_i < -2\varepsilon$, while independence forces every neighboring radius to be one. Thus $A_i - A_j < -2\varepsilon$, in the interior of the constant-one region of every factor of χ_i. The same holds near a zero coordinate because its neighbors are units. The gap between these cutoff thresholds, together with compactness, proves equality of the germs.

Let $\mathbf{1}_M \in \mathbb{Z}^V$ be the indicator vector of the mutable vertices and set

$$\vartheta = B\theta - \pi \mathbf{1}_M \pmod{2\pi}.$$

Unimodularity makes this an integral torus isomorphism, and $\omega = \sum_v dA_v \wedge d\vartheta_v$. On the seed-torus part of K, the support is

$$A_i \leq 0, \quad \vartheta_i = 0 \quad (i \in I), \qquad A_j = 0 \quad (j \notin I), \qquad I \subset M \text{ independent.} \tag{5.5}$$

This is the coordinate FLTZ germ for the fan of all independent cones, including cones of dimension greater than one.

Proposition 5.2. *Let $C \subset M$ be independent. A neighborhood of the entire deep C-cap chamber, including its centers and all residual conormal branches, has real exact cotangent coordinates in which the support is a product of $|C|$ disk zero sections and the residual coordinate FLTZ support. These coordinates preserve the remaining radial coordinates A_j used for independent collar cuts.*

Proof. Put $L = V \setminus C$, $P = B^{-1}$, and

$$\Gamma = B_{L,C}, \qquad R = P_{C,L}, \qquad E = 1 - \Gamma R.$$

The block identities in $PB = BP = 1$, together with $B_{C,C} = 0$, give

$$R\Gamma = 1, \qquad E^2 = E, \qquad RE = 0, \qquad P_{L,L}\Gamma = 0. \tag{5.6}$$

Use small real branches

$$t = \vartheta_C, \quad s = (BA)_C, \quad q = \vartheta_L, \quad \phi = Rq, \quad q_0 = Eq.$$

Since $q = \Gamma\phi + q_0$ and $s = -\Gamma^T A_L$,

$$A^T d\vartheta = \sum_{c \in C} (A_c\, dt_c - s_c\, d\phi_c) + A_L^T\, dq_0. \tag{5.7}$$

The residual angle torus is $\ker R$. Its moment covector is represented by $p_0 = A_L + R^T s$, which annihilates Γ.

Let $N(C)$ be the mutable neighbors of C and put $R_C = M \setminus (C \cup N(C))$. For $j \in R_C$, the j-row of Γ is zero, so $(q_0)_j = \vartheta_j$. The integral sections Ee_j show that these form a primitive subsystem of residual angle coordinates. Their dual momenta are

$$p_j = A_j - \sum_{c \in C} P_{jc} s_c.$$

The integral cotangent base shear

$$\phi'_c = \phi_c + \sum_{j \in R_C} P_{jc} \vartheta_j, \qquad p'_j = p_j + \sum_{c \in C} P_{jc} s_c = A_j \tag{5.8}$$

preserves the cotangent primitive. Complete these sections to an integral basis of the passive residual torus. Set

$$u_c = \exp(A_c + i\phi'_c), \qquad \xi_c = (s_c + it_c)/u_c. \tag{5.9}$$

On the punctured chart,

$$\begin{aligned} -\operatorname{Im}(\xi_c\, du_c) &= -t_c\, dA_c - s_c\, d\phi'_c, \\ A_c\, dt_c - s_c\, d\phi'_c &= -\operatorname{Im}(\xi_c\, du_c) + d(A_c t_c). \end{aligned}$$

Now $s_c + it_c = \mathcal{Q}_c$, and u_c/a_c is a smooth modulus-one unit depending only on unit coordinates and the chosen branches. Hence ξ_c extends across $a_c = 0$.

The inverse can be written without the phase of a vanishing variable. The residual momenta and s recover A_L. By (5.6),

$$\theta_L = P_{L,C} t + P_{L,L} q_0 + \text{constant};$$

the term involving ϕ cancels. Thus Q_c and u_c/a_c are already known, and

$$a_c = \frac{u_c}{u_c/a_c}, \qquad b_c = Q_c \frac{1 - \exp(u_c \xi_c)}{u_c} (u_c/a_c) \tag{5.10}$$

is smooth at $u_c = 0$. These are actual coordinates, and

$$\omega = \sum_{c \in C} \operatorname{Im}(du_c \wedge d\xi_c) + \omega_0.$$

On a sufficiently small neighborhood of the compact chamber support, the adjacent corrections in (5.4) vanish and every remaining active correction is in its linear angular branch. The primitive becomes

$$\sum_{c \in C} \{-\operatorname{Im}(\xi_c\, du_c) - \kappa\, dt_c\} + \lambda_0 - d \sum_{j \in R_C} h(p'_j) \vartheta_j.$$

The displayed exact corrections have potentials vanishing on the support, so they can be removed in contactification while fixing its lift. On the support, $\xi_c = 0$, all passive momenta vanish, and the remaining momenta are exactly $p'_j = A_j$. Equation (5.5) proves the assertion for the entire residual support. □

Lemma 5.3. *Let $\Lambda \subset T^*Q$ be fiberwise conic and contain the zero section. An exact symplectic germ Φ preserving Λ and the zero section admits a multiplicative deformation to the cotangent lift of its zero-section base map:*

$$\Phi_t = \delta_t^{-1} \Phi \delta_t, \qquad \delta_t(q,p) = (q, tp), \qquad 0 < t \le 1.$$

The family extends smoothly to $t = 0$. If $\Phi^* \lambda_{\mathrm{can}} - \lambda_{\mathrm{can}} = dF$ with $F = 0$ on the zero section, its primitive correction is $t^{-1} F(q, tp)$, which is also smooth at zero.

Proof. Write $\Phi = (Q, P)$ and let g be its zero-section map. Then $Q(q, 0) = g(q)$ and $P(q, 0) = 0$. Symplecticity gives $d_p P(q, 0) = (Dg(q))^{-T}$. Taylor expansion proves that the limit is the cotangent lift of g. Moreover, dF vanishes at the zero section, so F has no fiber terms of degree zero or one. This proves the assertion about the primitive correction. Conicity gives preservation of Λ, and conjugation is multiplicative. □

Proposition 5.4. *The cap charts admit a toric–cap Maslov datum for which their restriction functors and all multiple-face comparisons are the coordinate toric normal functors and disk-boundary functors. The circle filled in direction i has lattice vector Be_i, and its dual character is q_i^{-1}.*

Proof. Construct the datum over $\mathbb{Z}$, and extend coefficients to $\mathbb{C}$ for the categorical calculation. Start with the cotangent datum in the central coordinates. At an elementary collar use the normal polarization rotation in (5.9), and extend it over the disk using those cotangent coordinates. On an independent multiple face, perform all rotations in the simultaneous chart of Proposition 5.2. They occur in separate factors.

After these rotations, two overlapping presentations preserve the same product zero section and conic support. Factor out their zero-section base map and apply Lemma 5.3. Normalized local contact-isotopy quantization, starting with the canonical graph kernel, identifies the remaining transition with that kernel. Uniqueness of normalized quantization and multiplicativity of the contraction identify compositions and their higher associativity, by the Maslov-marked contact-germ invariance and descent constructions of [5, 9]. Potentials on triple overlaps telescope as differences of functions. Permuting independent rotations gives the graded tensor symmetry.

The inverse formula uses $P_{L,L}\Gamma = 0$, so no collapsed-phase action remains on the residual factor. A residual unit multiplying u_c gives a disk-base change and its cotangent lift. A full rotation of that real disk base has zero Maslov winding for the cotangent lift. Boundary sign choices extend simultaneously because the columns Be_i are part of an integral basis.

Finally, the original a_i-phase circle maps under $\vartheta = B\theta - \pi \mathbf{1}_M$ to Be_i, proving the character assertion. The boundary monodromy condition becomes $q_i = -1$ after a single torus sign rescaling: the mutable rows of the unimodular matrix B give a surjective map of sign tori. This rescaling preserves every coordinate boundary stratum. It fixes an unmarked mirror identification and need not preserve the originally named frozen-parameter map. □

Remark 5.5. Propositions 5.2 and 5.4 concern symplectic and contact germs and their microlocal categories. The global comparison in Section 7 applies GPS to a compatible hypersurface realization of the entire core; these local calculations alone do not identify the convex completion of a cut domain with a stopped cotangent model.

6 The microlocal category and its algebraic model

We keep the hypotheses and coordinates of Section 5. All categories are enhanced, all tensor products and pullbacks are derived, and all limits and colimits are homotopy limits and colimits. We work over $\mathbb{C}$. Superscript $\diamond$ denotes the large category; superscript w denotes its categorical compact objects, not the objects of finite microlocal rank. Write

$$A_0 = \mathbb{C}[z_i \ (i \in M), z_f^{\pm 1} \ (f \in F)].$$

Let Σ be the coordinate fan whose cones are indexed by independent subsets of M. Its toric variety is

$$X_\Sigma = \operatorname{Spec} A_0 \setminus \bigcup_{\{i,j\} \text{ mutable edge}} V(z_i, z_j). \tag{6.1}$$

For independent $S \subset M$, put $D_S = \{z_s = 0 : s \in S\}$ inside X_Σ. Every neighbor of S is a unit on D_S; in particular all $q_s = P_s/Q_s$, $s \in S$, are well-defined unit characters there.

Theorem 6.1. *For the toric–cap Maslov datum η of Proposition 5.4, there are equivalences*

$$\mu\mathrm{Sh}^{\diamond}_K(K;\eta) \simeq \mathrm{QCoh}(Y), \qquad \mu\mathrm{Sh}^{w}_K(K;\eta) \simeq \mathrm{Perf}(Y) = D^b\,\mathrm{Coh}(Y).$$

They concern the specified support in its specified symplectic germ. They do not require a comparison with the ordinary affine Liouville structure or an identification of completed Liouville sectors.

6.1 Coordinate toric categories and their normal functors

We give the special coherent–constructible correspondence needed here directly, including its functors on unbounded objects. The resulting equivalence is the coordinate-arrangement case of the nonequivariant correspondence of [7].

Stratify S^1 by one point and its complement. A weakly constructible sheaf is a diagram $W \rightrightarrows V$ of complexes. Allowing the zero section and one chosen conormal ray makes one generization an equivalence. Identifying W with V leaves one endomorphism $z : V \to V$. Consequently

$$\mathrm{Sh}^{\diamond}_{0\cup\mathbb{R}_{\le 0}}(S^1) \simeq \mathrm{Mod}_{\mathbb{C}[z]}\,. \tag{6.2}$$

We choose the normal microstalk convention

$$\mu(V) = \mathrm{cofib}(z : V \to V) = \mathbb{C} \otimes^{\mathbb{L}}_{\mathbb{C}[z]} V.$$

On the locally constant locus z is invertible. Frozen circles therefore contribute Laurent variables. Taking products gives Mod_{A_0} for the full coordinate orthant fan. This is the homotopy-coherent action of the free commutative monoid, equivalently a module over its monoid algebra.

For functions $f_1, \dots, f_s$ write

$$K(f_1, \dots, f_s) = \Big(A_0 \otimes \bigwedge(\eta_1, \dots, \eta_s), |\eta_i| = -1,\ d\eta_i = f_i\Big)\,.$$

The microstalk of the S-normal face is $K(z_s : s \in S) \otimes^{\mathbb{L}}_{A_0} E$. Removing the nonindependent cones consequently imposes precisely

$$K(z_i, z_j) \otimes^{\mathbb{L}}_{A_0} E = 0 \quad \text{for every mutable edge } \{i, j\}. \tag{6.3}$$

Larger missing faces vanish by tensoring additional Koszul factors.

Lemma 6.2. *Under (6.2) and its product version, condition (6.3) identifies the full category with* $\mathrm{QCoh}(X_\Sigma)$. *The normal microstalk along D_i is $\mathbb{L}i_i^*$, where $i_i : D_i \hookrightarrow X_\Sigma$. Its left adjoint is $\mathbb{R}i_{i*}[-1]$.*

Proof. For one edge $\{i, j\}$, the localization away from $V(z_i, z_j)$ is the augmented Čech functor

$$E \longmapsto [E[z_i^{-1}] \oplus E[z_j^{-1}] \longrightarrow E[(z_i z_j)^{-1}]],$$

in degrees zero and one. The fiber of its augmentation is local cohomology, expressed as a filtered colimit of dual Koszul complexes for (z_i^n, z_j^n), tensored with E. Those finite complexes belong to the thick tensor ideal generated by $K(z_i, z_j)$: the quotients by powers have finite filtrations by $A_0/(z_i, z_j)$, and the indicated sequences are regular. Thus (6.3) kills that fiber. Conversely tensoring the Čech complex by $K(z_i, z_j)$ kills all its terms. Tensoring the finitely

many edge localizations identifies their simultaneous essential image with the derived direct image of $\mathrm{QCoh}(X_\Sigma)$. This argument is valid for unbounded modules.

The i-normal microstalk is $K(z_i) \otimes^{\mathbb{L}}_{A_0} E$. For a neighbor j, condition (6.3) says that multiplication by z_j is an equivalence on this microstalk. All residual forbidden-face conditions remain. This is exactly the category on D_i, with the functor $\mathbb{L}i_i^*$. The statement is compatible with the Čech presentations because the normal Koszul complex is perfect.

The principal divisor has its specified normal coordinate z_i. For arbitrary complexes,

$$\mathbb{R}\operatorname{Hom}_{A_0}(A_0/(z_i), E) \simeq (A_0/(z_i) \otimes^{\mathbb{L}}_{A_0} E)[-1].$$

This identifies the left adjoint of $\mathbb{L}i_i^*$ with $\mathbb{R}i_{i*}[-1]$. Without the chosen normal trivialization it is $\mathbb{R}i_{i*}(- \otimes N_{D_i/X_\Sigma})[-1]$. □

In this presentation $\mathcal{O}_{X_\Sigma}$ is represented by its derived Čech direct image, not generally by the polynomial module A_0. For example $\mathbb{A}^2 \setminus \{0\}$ gives $[A_x \oplus A_y \to A_{xy}]$; the polynomial module has a nonzero forbidden-corner microstalk.

The other elementary functor is equally explicit. The large category on a disk zero section is $\mathrm{Mod}_{\mathbb{C}}$; on its annular collar it is $\mathrm{Mod}_{\mathbb{C}[w^{\pm 1}]}$. Restriction is inflation along $w = 1$, with left adjoint

$$E \longmapsto E \otimes^{\mathbb{L}}_{\mathbb{C}[w^{\pm 1}]} \mathbb{C}.$$

Inflation here is the actual derived restriction-of-scalars functor; it is not fully faithful. In the cap marking the boundary character is q_i^{-1}, so this left adjoint is specialization at $q_i = -1$. In particular it retains the full Koszul complex for $1 + q_i$.

6.2 The open cover and its entire enhanced diagram

Choose $\delta > 0$ sufficiently small in the linear end of h, and $0 < \varepsilon \ll \delta$. For independent $C \subset M$ put

$$V_C = \{p \in K : |a_i| < \delta + \varepsilon \ (i \in C), |a_i| > \delta - \varepsilon \ (i \notin C)\}.$$

These are open subsets of K and cover it. A nonempty finite intersection is labeled by a pair (J, C) of disjoint subsets, where $J \cup C$ is independent. Here C is deep in every chamber and J lies in the annular overlap. Define

$$X_{J,C} = D_{J \cup C} \cap \bigcap_{c \in C} \{q_c = -1\}, \qquad X_{\varnothing,\varnothing} = X_\Sigma. \tag{6.4}$$

The equations $q_c = -1$ are primitive independent equations in the unit-coordinate torus. Thus these spaces are smooth and quasi-compact.

The face category $\mathcal{P}$ has generating arrows

$$(J, C) \longrightarrow (J \setminus \{i\}, C), \qquad (J, C) \longrightarrow (J \setminus \{i\}, C \cup \{i\}), \quad i \in J, \tag{6.5}$$

with the independent-cube relations. It is the face category of the cubical complex formed by the cubes $[0, 1]^S$, S independent, with all other coordinates zero.

Here is the comparison with the actual Čech indexing category. A finite family $\mathcal{A}$ of chambers is sent to

$$C = \bigcap_{A \in \mathcal{A}} A, \qquad J = \left(\bigcup_{A \in \mathcal{A}} A \right) \setminus C.$$

With families ordered by reverse inclusion, the assignment $(J, C) \mapsto \{C \cup L : L \subset J\}$ is left adjoint to this functor. Adjoining those intermediate chambers does not change the intersection. The functor is therefore final for the corestriction colimit. This uses descent for an open cover, not for the closed cap cells.

Proposition 5.2 identifies the germ of the (J, C)-intersection with

$$0_{(D^2)^{|C|}} \times 0_{\mathbb{R}^{|J|}} \times \Lambda_{X_{J,C}}.$$

The intervals are the radial pieces of annular collars; their angular circles remain in $X_{J,C}$. Finite truncation of a conormal ray does not change these sections: the radial part of the microlocal category is locally constant under cotangent dilation, and its zero-section attachment is unchanged. Consequently the calculations above identify the large category of this intersection with $\mathrm{QCoh}(X_{J,C})$.

In the raw identifications, the two restriction functors opposite to (6.5) are

$$\mathbb{L}i_i^*, \qquad \mathbb{R}j_{i*},$$

where

$$\begin{aligned} i_i &: X_{J,C} \hookrightarrow X_{J\setminus\{i\},C}, && z_i = 0, \\ j_i &: X_{J\setminus\{i\},C\cup\{i\}} \hookrightarrow X_{J,C}, && 1 + q_i = 0. \end{aligned}$$

Their left adjoints are $\mathbb{R}i_{i*}[-1]$ and $\mathbb{L}j_i^*$. Normalize the equivalence at a vertex by the shift $s(J, C) = -|J| - |C|$. For the first arrow,

$$-1 + s(J \setminus \{i\}, C) - s(J, C) = 0,$$

and for the second arrow the difference of the vertex shifts is zero. The normalized corestriction diagram is therefore

$$\mathcal{D}_{J,C} = \mathrm{QCoh}(X_{J,C}), \qquad \mathbb{R}i_{i*},\ \mathbb{L}j_i^*. \tag{6.6}$$

This identification includes the transformations on multiple faces. To see this directly, choose a global ordering of the vertices and use the ordered Koszul models. Their comparison maps are

$$\begin{aligned} K(f) \otimes K(g) &\simeq K(f, g), && \eta_f \eta_g = -\eta_g \eta_f, \\ K(uf) &\longrightarrow K(f), && \eta_{uf} \longmapsto u\eta_f. \end{aligned}$$

For independent indices, neither normal coordinate occurs in the other exchange character, so the mixed squares are transverse derived base-change squares. Character changes are based integral lattice maps, whose group-algebra maps compose literally. For the signed exchange equations, changing $P_i + Q_i$ to $1 + q_i$ uses the unit Q_i on the relevant normal stratum and precisely the displayed unit-change map. The finite Čech localizations commute with these operations.

All higher comparisons are the associativity, graded permutation, and unit-change maps of the one ordered tensor complex. Their pentagon and signed-permutation identities give the higher coherences. Proposition 5.4 ensures that these are the transformations of the actual contact-germ identifications, rather than independently chosen algebraic replacements of the individual arrows.

The restrictions preserve colimits and have the indicated left adjoints. Microlocal sheaf descent can therefore be written as the colimit of this left-adjoint diagram:

$$\mu\mathrm{Sh}^{\diamond}_K(K; \eta) \simeq \operatorname*{colim}^{\mathrm{Pr}^L}_{(J,C)\in\mathcal{P}} (\mathrm{QCoh}(X_{J,C}), \mathbb{R}i_*, \mathbb{L}j^*). \tag{6.7}$$

Equivalently it is the limit of the right-adjoint restriction diagram. This is the large-category form of wrapped cosheaf descent [8, Section 3.6]; the passage to compacts will be made only after the large colimit is identified.

6.3 The algebraic colimit

Proposition 6.3. *The colimit in* (6.7) *is* $\mathrm{QCoh}(Y)$. *Its compact-object version is*

$$\operatorname*{colim}_{(J,C)\in\mathcal{P}} \big(\mathrm{Perf}(X_{J,C}), \mathbb{R}i_*, \mathbb{L}j^*\big) \simeq \mathrm{Perf}(Y).$$

Proof. For a smooth variety X, a smooth divisor $i : D \hookrightarrow X$, and a smooth divisor $j : H \hookrightarrow D$, the general blowup/delete theorem of [2, Theorem 4.13] gives

$$\mathrm{Perf}(X) \coprod_{\mathrm{Perf}(D)} \mathrm{Perf}(H) \simeq \mathrm{Perf}(\mathrm{Bl}_H X \setminus \widetilde{D}) \tag{6.8}$$

with arrows $\mathbb{R}i_*$ and $\mathbb{L}j^*$. All spaces are smooth here, so the theorem's coherent categories equal their perfect categories. We use this general algebraic theorem, not a geometric theorem identifying a truncated cluster mirror.

Apply (6.8) successively to

$$D_i = \{z_i = 0\} \subset X_\Sigma, \qquad H_i = \{z_i = 0,\ P_i(z) + Q_i(z) = 0\} \subset D_i.$$

On D_i, Q_i is a unit and $H_i = \{q_i = -1\}$. Adjacent divisors are disjoint. On an independent intersection the characters q_i form a primitive independent subsystem of the unit torus. The centers consequently become the independent coordinate pairs $(z_i, 1+q_i)$, so their modifications commute in these product charts. Their gluing uses the original z-coordinates.

On the chart with $z_l \neq 0$ for $l \notin I$, the retained affine modifications have equations

$$z_i w_i = P_i(z) + Q_i(z) \qquad (i \in I).$$

The other w_l are determined. Conversely every zero set in the full affine exchange model is independent by Lemma 5.1. These charts therefore cover exactly the full Y; no deep stratum has been discarded.

Fix an order on M for the successive pushouts. At each step also expand its divisor and center terms by all previously performed transverse modifications. The three states of an index are ambient, normal divisor, and imposed center. They are exactly the states outside $J \cup C$, in J, and in C of (6.4). A nonindependent state is empty. Transverse base change identifies the squares with the ordered Koszul comparisons already used in (6.6). Induction on the processed indices and associativity of colimits give the stated perfect-category colimit.

Its cocone can be specified explicitly. Put $S = J \cup C$ and

$$Z_S = \{a_s = 0 : s \in S\} \subset Y, \qquad \iota_S : Z_S \hookrightarrow Y.$$

The original coordinates give $p_{J,C} : Z_S \to X_{J,C}$, since the exchange equations impose $q_s = -1$ for every $s \in S$. The cocone is

$$\Psi_{J,C} = \mathbb{R}\iota_{S*}\mathbb{L}p_{J,C}^*. \tag{6.9}$$

The node charts show that Z_S is smooth and that its defining a_s form a regular sequence. For a toric arrow, its normal coordinate pulls back to that regular a_i, giving Tor-independent base change. For a cap arrow, $p_{J,C}$ factors through the character specialization. Ordered Koszul models identify their higher transformations. These are exactly the pullback and exceptional-divisor cocones in (6.8).

Finally Ind preserves these colimits in the category of compactly generated presentable stable categories with compact-preserving left adjoints. On these smooth quasi-compact varieties the Ind-completions are QCoh, proving the large version. □

Proof of Theorem 6.1. Combine (6.7) with Proposition 6.3. At each vertex the large category is compactly generated. A regular closed-divisor pushforward preserves perfect complexes, by its finite normal resolution, and derived pullback preserves perfect complexes. Thus the entire left-adjoint diagram preserves compacts. Taking compacts after the large equivalence gives the asserted wrapped microlocal equivalence. Smoothness of Y gives $\mathrm{Perf}(Y) = D^b\,\mathrm{Coh}(Y)$. □

Corollary 6.4. *The corestriction of the central object corresponding to $\mathcal{O}_{X_\Sigma}$ is a compact generator E, with*

$$\mathbb{R}\,\mathrm{Hom}(E,E) \simeq \frac{\mathbb{C}[a_i, b_i\ (i \in M), a_f^{\pm 1}\ (f \in F)]}{(a_i b_i - P_i - Q_i)_{i \in M}}.$$

Proof. The explicit root cocone in (6.9) sends the indicated object to $\mathcal{O}_Y$. The claim follows because Y is affine. □

6.4 A second algebraic argument by central localization

The following argument also identifies the colimit and explains why freezing is intrinsic to the computed diagram. Every vertex maps to $\mathrm{Spec}\,A_0$. Projection formula makes both i_* and j^* A_0-linear; the Koszul comparison maps are also A_0-linear. Hence the colimit $\mathcal{C}$ inherits a coherent A_0-action before any identification with $\mathrm{QCoh}(Y)$.

For $E \subset M$ let

$$P_E = A_0[z_E^{-1}] \otimes_{A_0} (-), \qquad z_E = \prod_{e \in E} z_e.$$

These are commuting exact reflective localizations, with coherent unit-compatible identifications $P_E P_{E'} = P_{E \cup E'}$. Localization commutes with the colimit and kills exactly the vertices for which $E \cap (J \cup C) \neq \varnothing$. The remaining diagram is the diagram with the variables in E frozen.

For an edge $\{i, j\}$, the Koszul object $K(z_i, z_j)$ acts by zero at every vertex and therefore on $\mathcal{C}$. If $P_i G = P_j G = 0$, put $H = K(z_j) \otimes G$. The equality $K(z_i) \otimes H = 0$ makes z_i invertible on H; but $P_i H = 0$, so $H = 0$. Thus z_j is invertible on G, and $P_j G = 0$ implies $G = 0$.

Lemma 6.5. *If P, Q are coherently commuting exact reflective localizations of a stable category with $\ker P \cap \ker Q = 0$, then*

$$\mathcal{C} \simeq \mathrm{Im}\,P \times_{\mathrm{Im}(PQ)} \mathrm{Im}\,Q.$$

Proof. Send $(A, B, QA \simeq PB)$ back to $A \times_{QA} B$ in $\mathcal{C}$. Exactness and commutation give its P- and Q-images as A and B. Conversely the fiber of $G \to PG \times_{PQG} QG$ is killed by both localizations. The two constructions are inverse as functors and hence also recover mapping complexes and higher coherence. □

For an edge, $Y = D(a_i) \cup D(a_j)$ because zero sets are independent. Freezing is precisely restriction to these opens. The lemma therefore agrees with QCoh Zariski descent. Induction on mutable edges terminates in products of node and torus models, giving another algebraic identification of the same computed category.

7 The global wrapped–microlocal comparison

The global input is [4, Corollary 7.22 and Remark 7.23]. Its regularity condition concerns the image of the relative core in a Liouville hypersurface embedding, together with boundary coordinates when a stop is present. We verify this condition for the actual primitive, construct compatible coefficient data, and use the cocores to obtain an equivalence.

7.1 An analytic contactification fixing the core

The cutoff functions in α need not be analytic. The following lemma verifies the embedded-core hypothesis of [4, Corollary 7.22].

Lemma 7.1. *Let $(X, \omega = d\alpha)$ be a finite-type Liouville manifold with compact core K. Suppose that X has a real-analytic structure, K is subanalytic singular isotropic, α restricts to zero on its smooth strata, and a neighborhood of K has an open cover on which*

$$\alpha = \lambda_a + dG_a, \qquad \lambda_a \text{ is real analytic}, \qquad G_a|_K = 0.$$

Then the contactification has, near $K \times \{0\}$, a real-analytic structure inducing its original smooth structure, in which $dt + \alpha$ is analytic and $K \times \{0\}$ is subanalytic. If X is stably polarized, it admits, after untwisted stabilization, a Liouville hypersurface embedding satisfying the embedded-core and boundary-coordinate hypotheses of [4, Corollary 7.22], compatible with the chosen stable polarization.

Proof. In the contactification use the coordinates $\tau_a = t + G_a(x)$. Then

$$dt + \alpha = d\tau_a + \lambda_a, \qquad d(G_b - G_a) = \lambda_a - \lambda_b.$$

A smooth function with analytic differential is locally analytic; this follows by integrating the differential in an analytic coordinate ball. Hence the changes $\tau_b = \tau_a + G_b - G_a$ are analytic. They define the required analytic contact atlas. In each chart the core lift is $\{x \in K, \tau_a = 0\}$, so is subanalytic.

Choose a compact Liouville domain inside the neighborhood on which this atlas is defined. Such a domain exists: sufficiently negative Liouville flow carries any fixed domain into any neighborhood of its core, since its backward images are nested compact sets with intersection K. The completion is unchanged.

By [4, Corollary 7.7], after stabilization there is a smooth Liouville hypersurface embedding in S^*M compatible with the polarization. A contact thickening is a codimension-zero isocontact embedding of a neighborhood in the contactification. The preceding atlas extends across the stabilization by taking G_a independent of its coordinates and using the standard analytic Liouville form there. For the stopped stabilization $(\mathbb{C}, \{\pm\infty\})^k$, the lifted relative core is $K \times \mathbb{R}^k \times \{0\}$ and is subanalytic in this atlas. Use a smoothed product Liouville domain whose boundary near the relative core is $|z| = R$, rounding the other corners away from that core. Its relative core has compact closure $K \times D^k_R$. We verify boundary coordinates in the contact-product sense of [4, Construction 7.14]. Write $z = q + ip$. The identity

$$\lambda_{\mathbb{C}^k} = \tfrac{1}{2}(q \cdot dp - p \cdot dq) = -p \cdot dq + d(q \cdot p/2)$$

is absorbed by replacing τ_a with $\widehat{\tau}_a = \tau_a + q \cdot p/2$. This gauge is zero on the relative core. Analytic cotangent polar coordinates $q = ru$, $u \in S^{k-1}$, give

$$dt + \alpha + \lambda_{\mathbb{C}^k} = -\xi_r \, dr + (d\widehat{\tau}_a + \lambda_a - \xi_u \, du).$$

Thus the ambient contact germ at the boundary is a product of the symplectic plane $T^*\mathbb{R}_r$ and the contactification of $X \times T^*S^{k-1}$. In it the relative core is

$$\{r < R, \xi_r = 0\} \times \{x \in K, \ u \in S^{k-1}, \ \xi_u = 0, \ \widehat{\tau}_a = 0\}.$$

After replacing r by $R - r$, these are analytic boundary coordinates of the stated form. There is no Reeb correction along the half-line, since $-\xi_r \, dr$ restricts to zero there. For $k = 0$ the unstopped boundary-coordinate condition is vacuous.

Apply [4, Lemma 7.4] to the contact thickening, with this analytic structure on its source. Over a compact neighborhood of the relative core it can be approximated by an analytic isocontact embedding. Its restriction to the original smooth hypersurface remains a Liouville hypersurface embedding: rescale the target contact form on the image so that its pullback is $dt + \alpha + \lambda_{\mathbb{C}^k}$ (or $dt + \alpha$ before stabilization). The embedded core and its boundary coordinates are subanalytic because the contact embedding is analytic in the new atlas and the relative core has compact closure. A sufficiently small approximation is homotopic to the original embedding through formal contact embeddings, so preserves the polarization data. □

Lemma 7.2. *The primitive α and core K constructed above satisfy the local hypotheses of Lemma 7.1.*

Proof. The manifold Y is real analytic and K is semialgebraic. The support calculation above proves stratumwise isotropy and the vanishing of α on those strata. On a seed-torus chart take $\lambda_a = A^T B\, d\theta$ and $G_a = -H$. Every summand of H vanishes on K: either its radial cutoff vanishes or its angular coordinate is zero.

At a point of K with zero-coordinate set I, use the neighborhood in Lemma 2.4. Its normal corrections have $h(A_i) = A_i + \kappa$, $\chi_i = \rho = 1$, and $\vartheta(T_i) = T_i$ for $i \in I$. Corrections at their neighbors vanish identically. Set

$$\lambda_a = A^T B\, d\theta - \sum_{i \in I} d((A_i + \kappa)T_i), \qquad G_a = -H_{\mathrm{res}},$$

where H_{res} is the sum of all remaining correction terms. It is smooth because its active indices and their neighbors have nonzero coordinates. The potentially singular part of λ_a is

$$-\sum_{i \in I} \mathrm{Im}\left(L_i \frac{da_i}{a_i} + \kappa\, dL_i\right), \qquad L_i = \mathrm{Log}(1 - a_i \widetilde{b}_i).$$

This extends real analytically: L_i/a_i has a convergent power series at $a_i = 0$. All remaining terms involve logarithms and angles of unit coordinates and are real analytic. Finally, every summand of H_{res} vanishes on the entire local support. If $A_j = 0$, then $h(A_j) = 0$; if $A_j < 0$, the support equations give $b_j = 0$ and $q_j = -1$, hence $T_j = 0$. These charts cover K and prove the claim for the actual global primitive. □

7.2 A compatible stable polarization

We construct the toric–cap datum over $\mathbb{Z}$ before extending coefficients to $\mathbb{C}$. This specifies the coefficient twisting used below. By [4, Section 5.3 and Remark 5.2], ambient grading/orientation data consist of a trivialization of $(\det_{\mathbb{C}} TY)^{\otimes 2}$ and a background map $Y \to K(\mathbb{Z}/2, 2)$. Once such data exist, their homotopy classes form a torsor for

$$H^1(Y; \mathbb{Z}) \oplus H^2(Y; \mathbb{Z}/2).$$

This statement concerns ambient data. The choices of brane structure on an unobstructed individual Lagrangian instead form a torsor for $H^0(L; \mathbb{Z}) \oplus H^1(L; \mathbb{Z}/2)$.

Lemma 7.3. *Let $T_0 \subset K_B$ be the central real torus. Restriction induces injections*

$$H^1(K_B; \mathbb{Z}) \longrightarrow H^1(T_0; \mathbb{Z}), \qquad H^2(K_B; \mathbb{Z}/2) \longrightarrow H^2(T_0; \mathbb{Z}/2).$$

Proof. Filter K_B by the unions K_s of the closed caps indexed by independent sets of size at most s. Thus $K_0 = T_0$. A cap indexed by C, with $|C| = s$, is $(D^2)^s \times T^{d-2s}$, and

its intersection with K_{s-1} is its disk-factor boundary. Different cap interiors are disjoint. Semialgebraic triangulation makes these inclusions cofibrations, so

$$H^j(K_s, K_{s-1}; A) \cong \bigoplus_{\substack{C \text{ independent} \\ |C|=s}} H^{j-2s}(T^{d-2s}; A).$$

For $s = 1$ the connecting map

$$H^1(T_0; A) \longrightarrow H^2(K_1, T_0; A) = A^M$$

evaluates classes on the circles filled by the elementary caps. These are the coordinate circles e_i in the original a-phases, or the columns Be_i in the central cotangent angles. Hence this map is surjective. The relative long exact sequence gives the claimed injectivity in degrees one and two for K_1. For $s \geq 2$ the relative groups in degrees at most three vanish, so the conclusion holds for K_B. □

Proposition 7.4. *The symplectic manifold $(Y_B, \operatorname{Im}\Omega)$ admits a stable polarization whose induced integral grading/orientation datum restricts on K_B to a datum homotopic to the toric–cap datum η. Consequently the same datum may be used in the GPS comparison and in Theorem 6.1 after extension of coefficients to $\mathbb{C}$.*

Proof. Put $m = |M|$. The presentation (2.1) exhibits Y_B as a smooth complete intersection in $\mathbb{C}^{2m} \times (\mathbb{C}^*)^{|F|}$ with trivial normal bundle. Its tangent exact sequence splits smoothly, giving

$$(TY_B, I) \oplus \mathbb{C}^m \cong \mathbb{C}^{d+m}.$$

The stable determinant is explicit. Give the ambient space its ordinary volume in the mutable (a_i, b_i) variables and its logarithmic volume in the frozen variables. The residue along the exchange equations has, on the seed torus, the expression

$$\nu_{\mathrm{res}} = \pm \bigwedge_{v \in V} d\log a_v.$$

Indeed, the normal Jacobian in the b_i variables is $\prod_{i \in M} a_i$. Thus ν_{res} differs by a nonzero constant from $\Omega^{d/2}/(d/2)!$.

Here I is the algebraic complex structure. To pass to a complex structure compatible with $\omega = \operatorname{Im}\Omega$, reduce the holomorphic symplectic frame bundle from $\mathrm{Sp}(d/2, \mathbb{C})$ to its maximal compact subgroup. The resulting almost hyperkähler structure contains I and an ω-compatible J. Choose this reduction constant in logarithmic coordinates near T_0. Such a prescribed local choice extends because the space of reductions is contractible. Since I and J anticommute,

$$Q = \frac{1 - JI}{\sqrt{2}}, \qquad QI = JQ,$$

identifies their complex tangent bundles. We therefore obtain a stable complex frame for (TY_B, J).

Let $\nu = \Omega^{d/2}/(d/2)!$, and transfer this nonvanishing determinant form by Q to a form ν_J for J. Multiplying one vector of the stable frame by a nowhere-zero function makes its determinant agree with ν_J and the standard stabilizing determinant. Unitary normalization changes only the positive modulus. The real span of the normalized frame is a global stable Lagrangian polarization $\mathcal{P}$. Its real bundle is trivial, so its background class $w_2(\mathcal{P})$ vanishes; its squared determinant grading is the one defined by $\nu_J^{\otimes 2}$.

On T_0 the form Ω, the chosen reduction, and Q are constant in the logarithmic tangent frame. The cotangent polarization in (A, ϑ) coordinates is constant in the same frame. Their

determinant phases consequently differ by a constant, and their grading data are homotopic. Both real bundles are trivial, so their orientation backgrounds agree as well.

The datum η restricts by construction to the cotangent datum on T_0. The difference between η and the datum induced by $\mathcal{P}$ therefore vanishes on T_0. Lemma 7.3 makes this difference zero on K_B. Choosing a homotopy of the restricted data and transporting the local identifications along it proves the claim.

The stable polarization was constructed first. Its comparison with η uses the coefficient map

$$U/O \longrightarrow B\operatorname{Pic}(\mathbb{Z}) = K(\mathbb{Z},1) \times K(\mathbb{Z}/2,2)$$

from [4, Remark 5.2], followed by base change. □

7.3 Homological cocores and the comparison functor

Proposition 7.5. *The Liouville manifold (Y_B, α) admits homological cocores in the sense of [4, Definitions 7.1–7.2].*

Proof. By Proposition 2.9, each smooth Lagrangian component of the core has a proper exact Lagrangian which is cylindrical at infinity and intersects the core once, transversely, at a smooth point. The observation immediately after [4, Definition 7.1] says precisely that such a Lagrangian is a homological cocore.

To cover every smooth point of the same component, choose a path inside the smooth locus. Its compact image has a neighborhood in which the core is a smooth Lagrangian. In successive cotangent charts, a compactly supported lift of a base isotopy moves the intersection along that path while preserving the core. The resulting Hamiltonian isotopy is compactly supported and preserves the single transverse intersection property. Thus the criterion holds at every smooth Lagrangian point, as required by Definition 7.2 of [4]. □

Theorem 7.6 (Wrapped HMS for the explicit primitive)**.** *Under the standing unimodular acyclic hypotheses, the compatible grading/orientation data constructed above give equivalences*

$$\operatorname{Perf}\mathcal{W}(Y_B,\alpha)^{\mathrm{op}} \simeq \mu\mathrm{Sh}^{w}_{K_B}(K_B;\eta) \simeq \operatorname{Perf}(Y_B) = D^b\operatorname{Coh}(Y_B).$$

Proof. Proposition 7.4 supplies a stable polarization and identifies its induced integral coefficient datum on K_B with η. Lemmas 7.1 and 7.2 give a compatible Liouville hypersurface embedding after untwisted stabilization, with subanalytic relative core and the required boundary coordinates. Thus [4, Corollary 7.22], applied as in the proof of its Theorem 1.4, gives a fully faithful functor

$$\operatorname{Perf}\mathcal{W}(Y_B,\alpha)^{\mathrm{op}} \longrightarrow \mu\mathrm{Sh}_{K_B}(K_B;\eta)^{c}.$$

Untwisted stabilization preserves the microlocal category; on the wrapped side its Künneth embedding is a Morita equivalence by Proposition 7.5 and [4, Proposition 7.3]. The comparison sends homological cocores to co-representatives of microstalks. These generate the compact microlocal category, so the functor is essentially surjective after pretriangulated idempotent completion. This is the homological-cocore clause of [4, Theorem 1.4]; no assertion that the explicit Liouville field is Morse-gradient-like is used. Finally apply Theorem 6.1. □

Corollary 7.7 (Ordinary affine form)**.** *Let β be the ordinary affine Kähler primitive obtained from any closed algebraic embedding of Y_B. With the transported grading/orientation data,*

$$\operatorname{Perf}\mathcal{W}(Y_B,\beta)^{\mathrm{op}} \simeq \operatorname{Perf}(Y_B).$$

Proof. Corollary 4.7 compares the wrapped categories of the explicit and ordinary affine structures. Compose that equivalence with Theorem 7.6. Lemma 3.10 handles the choice of closed algebraic embedding. □

The equivalence uses the sign and character changes in Proposition 5.4 and a chosen homotopy of the coefficient data. It is an unmarked equivalence; compatibility with the original frozen-parameter map, a specified Calabi–Yau enhancement, and independence of every auxiliary embedding choice are not asserted.

8 Finite-type open Grassmannians

Let

$$U_{k,n} = \{V \in \mathrm{Gr}(k,n) : p_{[i,i+k-1]}(V) \neq 0 \text{ for every cyclic interval } [i, i+k-1]\}.$$

We use the projective Grassmannian, with its frozen coordinates retained. On the chart $p_{123} \neq 0$, the notation p_I below means p_I/p_{123} for a Plücker coordinate; a homogeneous cluster variable of degree e is similarly divided by p_{123}^e.

Scott's coordinate-ring theorem and his bipartite Dynkin seeds identify the localized homogeneous coordinate rings in the three cases $(k,n) = (3,6), (3,7), (3,8)$ [10]. Passing to $p_{123} = 1$ gives their projective affine coordinate rings. The acyclic presentation of [1, Corollary 1.21] applies over the Laurent polynomial ring in the remaining frozen coordinates. Thus the seed variables and their first mutation partners generate the whole coordinate ring, and the exchange relations generate its defining ideal. This identifies the entire affine variety, rather than only a seed chart.

8.1 The two-parameter D_4 family

Put

$$(f_1, f_2, f_3, f_4, f_5, f_6) = (p_{123}, p_{234}, p_{345}, p_{456}, p_{156}, p_{126}), \qquad f_1 = 1,$$

and set

$$\begin{aligned} (L_1, L_2, L_3) &= (p_{145}, p_{136}, p_{235}), \\ (L_1', L_2', L_3') &= (p_{236}, p_{245}, p_{146}), \qquad Y = L_1 L_1' - f_1 f_4. \end{aligned}$$

Scott's D_4 seed consists of L_1, Y, L_2, L_3; the corresponding first mutation partners are $L_1', p_{135}, L_2', L_3'$. Its exchange relations are

$$\begin{aligned} L_1 L_1' &= Y + f_1 f_4, \quad L_2 L_2' = Y + f_3 f_6, \\ L_3 L_3' &= Y + f_2 f_5, \quad Y p_{135} = L_1 L_2 L_3 + f_1 f_3 f_5. \end{aligned} \tag{8.1}$$

These identities can also be checked directly by expanding the minors of $[I_3 \mid A]$.

Proposition 8.1. *There is an algebraic isomorphism*

$$U_{3,6} \cong Y_D \times (\mathbb{C}^*)^3,$$

where

$$Y_D = \left\{ \begin{array}{ll} z y_0 = 1 + x_1 x_2 x_3, & x_1 y_1 = 1 + z, \\ x_2 y_2 = 1 + t z, & x_3 y_3 = 1 + s z, \end{array} \quad t, s \neq 0 \right\}. \tag{8.2}$$

The three passive coordinates can be taken to be f_2, f_3, f_4.

Proof. Let $C = f_3 f_5$. Define

$$\begin{aligned} z &= Y/f_4, & x_1 &= L_1/C, & x_2 &= L_2, & x_3 &= L_3, \\ y_0 &= p_{135} f_4/C, & y_1 &= L_1' C/f_4, & y_2 &= L_2'/(f_3 f_6), & y_3 &= L_3'/(f_2 f_5), \\ t &= f_4/(f_3 f_6), & s &= f_4/(f_2 f_5). \end{aligned}$$

Substitution in (8.1) gives (8.2). Every denominator is a frozen unit. The change of frozen variables has inverse

$$f_5 = \frac{f_4}{sf_2}, \qquad f_6 = \frac{f_4}{tf_3}.$$

All remaining changes are invertible rescalings by units. The acyclic presentation therefore makes this change an isomorphism of coordinate rings, not only of their fraction fields. □

The complete graph of (8.2) has edges

$$\{z, x_1\}, \quad \{z, x_2\}, \quad \{z, x_3\}, \quad \{x_2, t\}, \quad \{x_3, s\}.$$

It is the E_6 tree with the two endpoints t, s frozen. In particular, its complete skew exchange matrix is unimodular, even though the mutable D_4 matrix alone is not.

For an explicit check of the higher intersections, write $N = \{(u, v) \in \mathbb{C}^2 : uv \neq 1\}$. There is an algebraic open cover

$$\begin{aligned} Y_D &= D(z) \cup D(x_1x_2x_3), \\ D(z) \cong N^3, &\qquad D(x_1x_2x_3) \cong N \times (\mathbb{C}^*)^4. \end{aligned} \tag{8.3}$$

Indeed, in the first chart take $u_j = x_j$, $v_j = y_j$, and $q_j = u_jv_j - 1$. The inverse is

$$z = q_1, \quad t = q_2/q_1, \quad s = q_3/q_1, \quad y_0 = (1 + u_1u_2u_3)/q_1.$$

In the second chart take $u = z$, $v = y_0$, and the four unit coordinates x_1, x_2, t, s. Then $q = uv - 1$ is nonzero, $x_3 = q/(x_1x_2)$, and the other partners follow from (8.2). The two opens cover because $z = 0$ implies $x_1x_2x_3 = -1$; their intersection is $(\mathbb{C}^*)^6$. We use (8.3) only as an algebraic cover.

The point represented by $[I_3 \mid I_3]$ belongs to $U_{3,6}$ and has

$$z = -1, \qquad t = s = 1, \qquad x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = 0, \qquad y_0 = -1.$$

Thus the triple leaf cap is present in the affine variety. A calculation retaining only single caps would omit this point.

8.2 Removing coefficients in types E_6 and E_8

For a finite bipartite graph Γ, let Q_Γ be its unsigned adjacency matrix, and define

$$X_\Gamma = \left\{ a_ib_i = 1 + \prod_{j \sim i} a_j \text{ for every vertex } i \right\}.$$

Lemma 8.2. *Suppose Q_Γ is unimodular. Over the Laurent coefficient ring $\mathbb{C}[f_1^{\pm 1}, \dots, f_h^{\pm 1}]$, every presentation*

$$x_iy_i = 1 + f^{C_i} \prod_{j \sim i} x_j$$

is isomorphic to the coefficient-free presentation of X_Γ tensored with this Laurent ring.

Proof. View C as the matrix whose ith row is C_i, and put $S = -Q_\Gamma^{-1}C$. Since Q_Γ is unimodular, every entry of S is an integer. Substitute

$$x_i = f^{S_i}a_i, \qquad y_i = f^{-S_i}b_i.$$

The coefficient in the ith relation becomes $f^{C_i + \sum_{j \sim i} S_j} = 1$. The substitution and its inverse use only integral powers of units, so both are regular. □

We fix the edge sets

$$E_6: \quad 01, 02, 23, 04, 45,$$
$$E_8: \quad 01, 02, 23, 04, 45, 56, 67.$$

The first tree has perfect matching $\{01, 23, 45\}$; the second has $\{01, 23, 45, 67\}$. A tree has at most one perfect matching, since the symmetric difference of two would contain a cycle. In the determinant expansion of a tree adjacency matrix, the nonzero terms are precisely its perfect matchings, each with sign $(-1)^{|V|/2}$. Consequently

$$\det Q_{E_6} = -1, \qquad \det Q_{E_8} = 1.$$

The bipartite skew matrices have determinant one in both cases.

Proposition 8.3. *There are algebraic isomorphisms*

$$U_{3,7} \cong X_{E_6} \times (\mathbb{C}^*)^6, \qquad U_{3,8} \cong X_{E_8} \times (\mathbb{C}^*)^7.$$

Proof. Use Scott's bipartite Dynkin seed and invert the cyclic frozen Plücker coordinates. Set $p_{123} = 1$. In each exchange relation, one monomial contains no mutable variable because the mutable vertex is a source or a sink. Divide its mutation partner by that frozen monomial. The relation now has the form of Lemma 8.2. Apply the lemma and the acyclic presentation. There remain respectively six and seven independent frozen units. □

8.3 Passive torus factors

The active factors Y_D, X_{E_6}, and X_{E_8} have complex dimensions 6, 6, and 8. The passive torus dimensions are 3, 6, and 7. On a passive coordinate $t = e^{T+i\theta}$ we use

$$\lambda_T = T\, d\theta.$$

This definition makes sense in every passive dimension. In particular, we do not assert a holomorphic symplectic form on all of $U_{3,6}$ or $U_{3,8}$, whose complex dimensions are odd.

The product Liouville form has skeleton equal to the active skeleton times the compact torus. This follows directly from its product flow: an orbit is trapped exactly when both factor orbits are trapped. There is also an explicit comparison with a split closed affine embedding. For $t \mapsto (t, t^{-1})$, the ordinary affine primitive is

$$\beta_T = \frac{1}{2}\operatorname{Im}\bigl(\bar{t}\, dt + \overline{t^{-1}}\, d(t^{-1})\bigr) = \sinh(2T)\, d\theta.$$

The diffeomorphism

$$(T, \theta) \longmapsto \left(\frac{1}{2}\operatorname{arcsinh} T, \theta\right)$$

pulls β_T back to λ_T. Taking products gives a strict exact comparison for the passive factor. Combined with an active-factor exact comparison, this yields an exact comparison with the split affine primitive on each of the three open Grassmannians above. Lemma 3.10 compares its wrapped category with that from any other closed algebraic embedding of the same variety. This is a separate Liouville-domain deformation.

Corollary 8.4. *For $n = 6, 7, 8$, and the ordinary affine Liouville structure on $U_{3,n}$ from any closed algebraic embedding, there are compatible grading/orientation data and an unmarked equivalence*

$$\operatorname{Perf} \mathcal{W}(U_{3,n})^{\mathrm{op}} \simeq \operatorname{Perf}(U_{3,n}) = D^b \operatorname{Coh}(U_{3,n}).$$

Proof. Apply Theorem 7.6 to the active factors in Propositions 8.1 and 8.3. A passive factor has the standard cotangent model T^*S^1, whose compact microlocal category is $\operatorname{Perf}\mathbb{C}[t^{\pm 1}]$. The active cocores and cotangent fibers give product homological cocores. The Künneth equivalence of [3, Theorem 1.5 and Corollary 1.18] therefore identifies the product wrapped category with the tensor product of these factors. On the algebraic side, the coordinate rings tensor with the Laurent polynomial ring. The explicit passive comparison, Corollary 4.7, and Lemma 3.10 give the ordinary affine form for any closed algebraic embedding. □

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